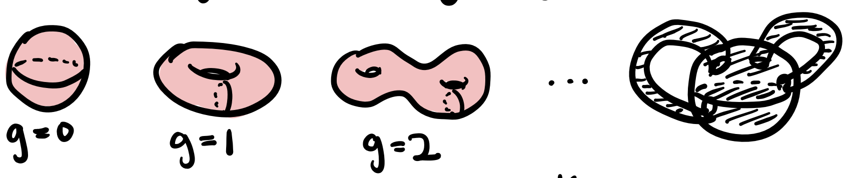
INTRO TO TRISECTIONS

all manifolds are smooth

2D: standard genus g surface $\Sigma_g = \#^g(S^1 \times S^1)$

3D: standard genus g handlebody $H_g = \natural^g(S^1 \times B^2)$ w/ $\partial H_g = \Sigma_g$



4D: standard 4D 1-handlebody $Z_k = \natural^k(S^1 \times B^3)$ w/ $\partial Z_k = \#^k(S^1 \times S^2)$

<u>Definition</u> [Gay+Kirby]: a $(g;k_1,k_2,k_3)$-trisection $\mathcal{T}$ of a closed, connected, oriented 4-mfld X is a decomposition $X = X_1 \cup X_2 \cup X_3$ such that:

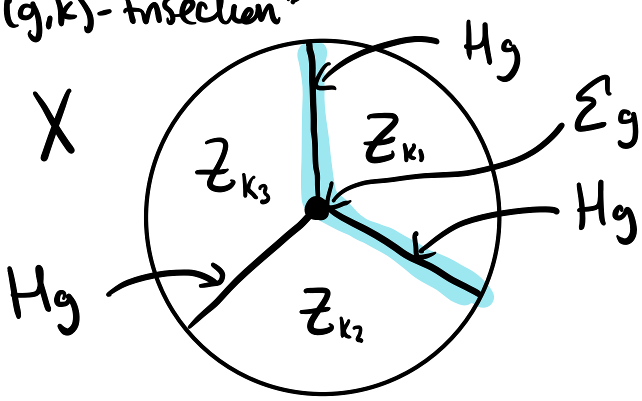
① for each i, $X_i \cong Z_{k_i}$

② consider $i \pmod 3$, $X_i \cap X_{i+1} \cong H_g$
(orient as a submfld of ∂X_{i+1})

③ $X_1 \cap X_2 \cap X_3 \cong \Sigma_g$
(orient as $\partial(X_1 \cap X_2) = \partial(X_2 \cap X_3) = \partial(X_3 \cap X_1)$)

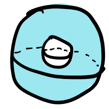
$\exists$ "relative trisections" for 4-mflds w/ ∂ [Castro, Gay, Pinzon-Caicedo]

a trisection is balanced if $k_1 = k_2 = k_3 = k$, in which case we write "(g,k)-trisection"



note: (1) this is 4D analog of a Heegaard splitting
(every 3-mfld is union of two handlebodies)

(2) $\underbrace{\partial Z_k}_{\text{3-mfld}} = H_g \cup H_g$ is a Heegaard splitting for $\partial Z_k = \#^k(S^1 \times S^2)$



$S^2 \times I$

(3) $\mathcal{T}$ is determined by the spine $= H_g \cup H_g \cup H_g$
(Laudenbach + Poenaru: can fill in ∂Z_k in a unique way)

Theorem [Gay + Kirby]:

Existence: every closed, connected, oriented 4-mfld has a trisection

uniqueness: any two trisections of the same 4-mfld become isotopic after number of stabilizations (later)

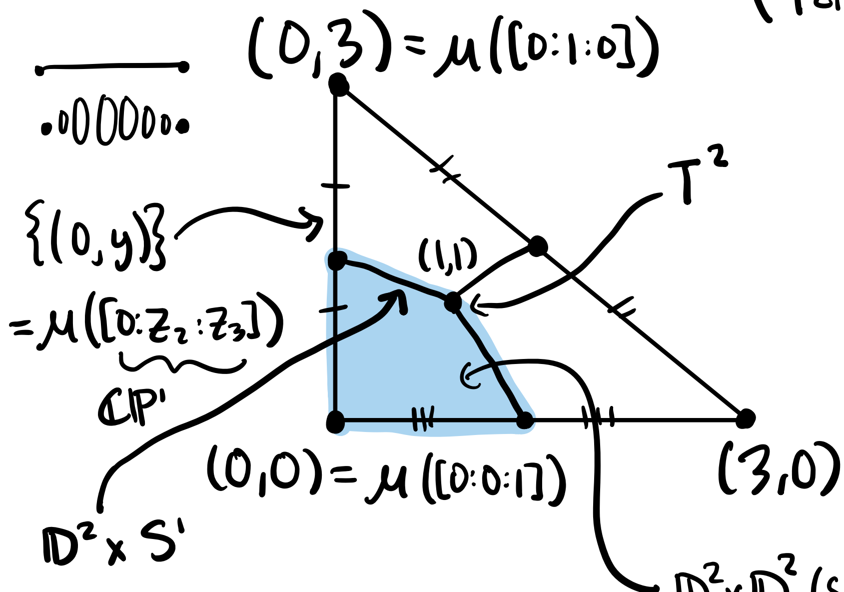
proof: ask Swapnanil
(trisecting Morse 2-function) $\square$

ex $\mathbb{CP}^2$ admits a (1,0)-trisection → used to reprove Thom conj [PLC]

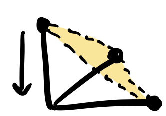
$$\mathbb{CP}^2 = \{[z_1 : z_2 : z_3] \mid z_i \in \mathbb{C},\ [z_1 : z_2 : z_3] = [\lambda z_1 : \lambda z_2 : \lambda z_3],\ \lambda \in \mathbb{C}\setminus\{0\}\}$$

moment map $\mu : \mathbb{CP}^2 \to \mathbb{R}^2$

$$\mu([z_1 : z_2 : z_3]) = \left(\frac{3|z_1|^2}{|z_1|^2 + |z_2|^2 + |z_3|^2},\ \frac{3|z_2|^2}{|z_1|^2 + |z_2|^2 + |z_3|^2} \right)$$



$\mu : \mathbb{CP}^2 \to \mathbb{R}^3$
image lies in the plane $x+y+z=3$



$= \mathbb{D} \times \mathbb{D}$ (smooth corners to get B^4)

$|z_1| = |z_2| = |z_3| = 1$

know $z_3 \neq 0$

assume $z_3 = 1$

$[z_1 : z_2 : 1]$

$|z_1| = |z_2| = 1$

$S^1 \times S^1 = T^2$

$y = -\frac{1}{2} x + \frac{3}{2}$

$x + 2y = 3$

$x \leq 1 \quad y \geq 1$

assume $z_3 = 1$

$$\begin{array}{ccc} |z_1| \leq 1 & & |z_2| \leq 1 \\ \downarrow & & \downarrow \\ & [z_1 : z_2 : 1] & \\ \uparrow & & \uparrow \\ |z_1| \leq 1 & & |z_2| = 1 \\ \mathbb{D}^2 & \times & S^1 \end{array}$$

$$\frac{3|z_1|^2}{|z_1|^2 + |z_2|^2 + 1} + 2\left(\frac{3|z_2|^2}{|z_1|^2 + |z_2|^2 + 1}\right) = 3$$

$$3|z_1|^2 + 6|z_2|^2 = 3|z_1|^2 + 3|z_2|^2 + 3$$

$$6|z_2|^2 = 3|z_2|^2 + 3$$

$$|z_2|^2 = 1$$

$\Sigma_g = T^2$

$H_g = \mathbb{D}^2 \times S^1$

$Z_k = B^4$

GROUP TRISECTIONS! [Abrams, Gay, Kirby]

a (g,k)-trisection of a group G is a commutative cube (van-Kampen) of groups as shown below, st every homomorphism is surjective and each face is a pushout

$$\begin{tikzcd}
& \pi_1(H_g) \arrow[r] \arrow[dr] & \pi_1(Z_k) \arrow[dr] & \\
\pi_1(\Sigma_g) \arrow[ur] \arrow[r] \arrow[dr] & \pi_1(H_g) \arrow[ur] \arrow[dr] & \pi_1(Z_k) \arrow[r] & \pi_1(X) = G \\
& \pi_1(H_g) \arrow[r] \arrow[ur] & \pi_1(Z_k) \arrow[ur] &
\end{tikzcd}$$

$\pi_1(\Sigma_g) = \langle a_1, b_1, \dots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] \rangle$

$\pi_1(H_g) = \langle x_1, \dots, x_g \rangle$

$\pi_1(Z_k) = \langle x_1, \dots, x_k \rangle$