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3·26·20

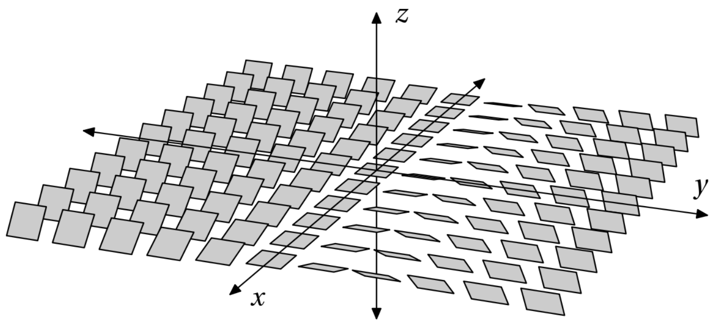
{ Chekanov's DGA for Legendrian Knots: why $d^2 = 0$ }

trisector jr. skype mtg notes

Ⓘ Background

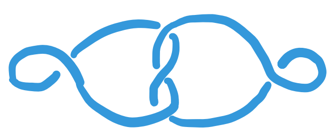
→ DGA's of Leg. iso. knots have same "stable type"

- **Chekanov (1997)**: an invariant of Legendrian knots using DGA's that can distinguish Legendrian knots w/ the same classical invt's
  (Eliashberg (1998): general contact homology ... Chekanov's work is a combinatorial version of this in a specific context)
- the **standard contact structure on $\mathbb{R}^3$** is the 2-plane distribution ξ_{std} which is locally the kernel of the 1-form $\alpha = dz - y\,dx$



twist along y-axis, no change along x- and z-axis

- a **Legendrian knot** L (in $(\mathbb{R}^3, \xi_{std})$) is a smooth knot whose tangent vectors are contained in the contact planes of ξ_{std}
  - in particular there are no vertical tangencies
- **Lagrangian projection** ("diagram") – $\pi : \mathbb{R}^3 \to \mathbb{R}^2$, $(x,y,z) \mapsto (x,y)$



right-handed trefoil

  - make all self intersections are transverse double pts
  - enclosed signed area is 0; recover z by integrating $dz = y\,dx$

II The DGA (differential graded algebra (A, ∂))

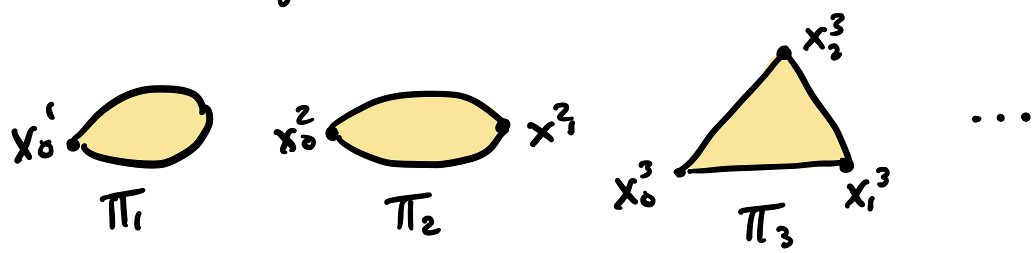
- **Algebra** – given Legendrian L, let $a_1, \dots, a_n$ be the crossings of $\pi(L)$.
Let $A :=$ free assoc. (unital) algebra over $\mathbb{Z}_2$ w/ generators $a_1, \dots, a_n$; aka algebra of non commutative polynomials in variables $a_1, \dots, a_n$ w/ coeff in $\mathbb{Z}_2$

- **Graded** – $\deg(a_i) \in \mathbb{Z}/m(L)\mathbb{Z}$
$m(L)$ ← maslov index of L

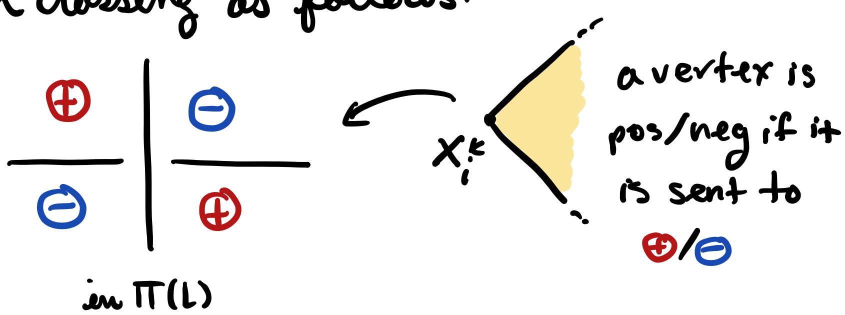
- **Differential** – in general, a graded linear map $\partial : A \to A$ of deg -1 ($\partial(A_i) \subseteq A_{i-1}$) st $\partial(ab) = \partial(a)b + a\partial(b)$ $\forall a, b \in A$ (Leibniz rule) and $\partial^2 = 0$

for our DGA:

- $\partial(a_i)$ is a count of certain immersed polygons w/ a vertex sent to a_i
- $\Pi_k :=$ convex k-gon $\subseteq \mathbb{R}^2$ w/ vertices $x_0^k, \dots, x_{k-1}^k$ (counter clockwise)



- consider smooth orientation-preserving immersions $f : \Pi_k \to \mathbb{R}^2$ st $f(\partial \Pi_k) \subset \pi(L)$. Furthermore, consider isotopy classes rel vertices. So, vertices go to crossings, edges go to edges in diagram.
- Label each crossing as follows:



- an *admissible immersion* is an immersion satisfying the above and st x_0^k is positive and all other vertices are negative

- $W_k(\Pi(L), a_i) :=$ set of all admissible immersions of Π_k that send x_0^k to a_i
- $\partial = \sum_{k \geq 0} \partial_k$ where

counter clockwise listing the neg crossings for each Π_{k+1}

$$\boxed{\partial_k(a_i) = \sum_{f \in W_{k+1}(\Pi(L), a_i)} f(x_1^{k+1}) \cdots f(x_k^{k+1})}$$

if $k=0$, above formula counts # of 1-gons (mod 2)

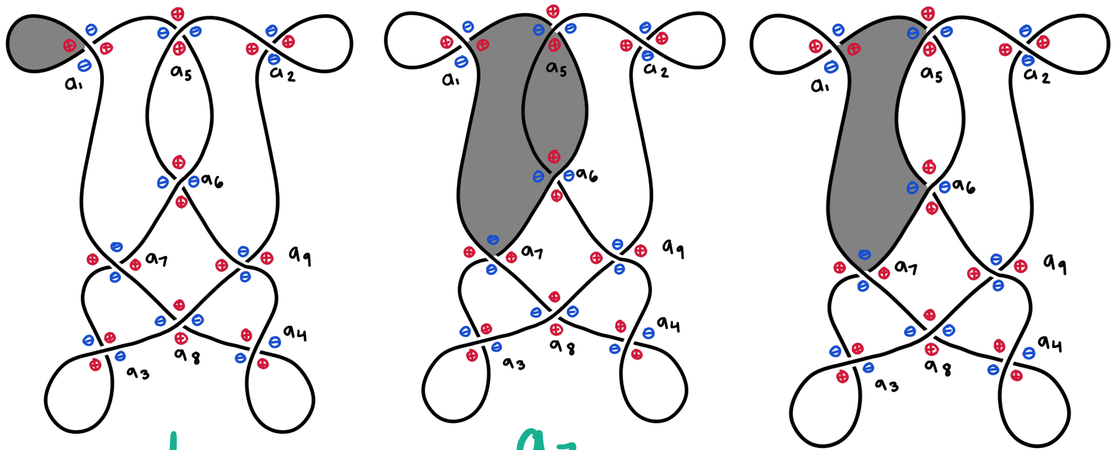
- extend ∂ to all of A by linearity and Leibniz rule

- **Lemma**: there are only finitely many admissible immersions
- **Lemma**: $\deg(\partial) = -1$
- **theorem**: $\partial^2 = 0$

Ex

calculating ∂ of a crossing

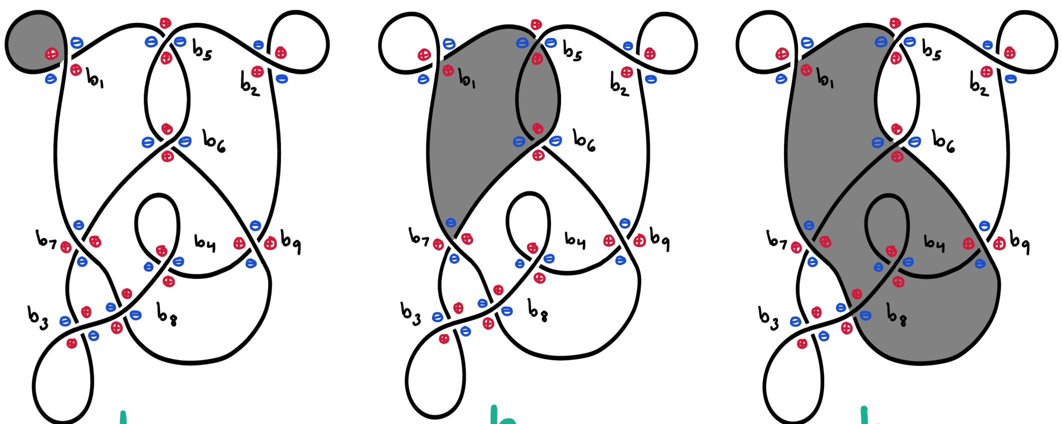
(A, ∂_A)



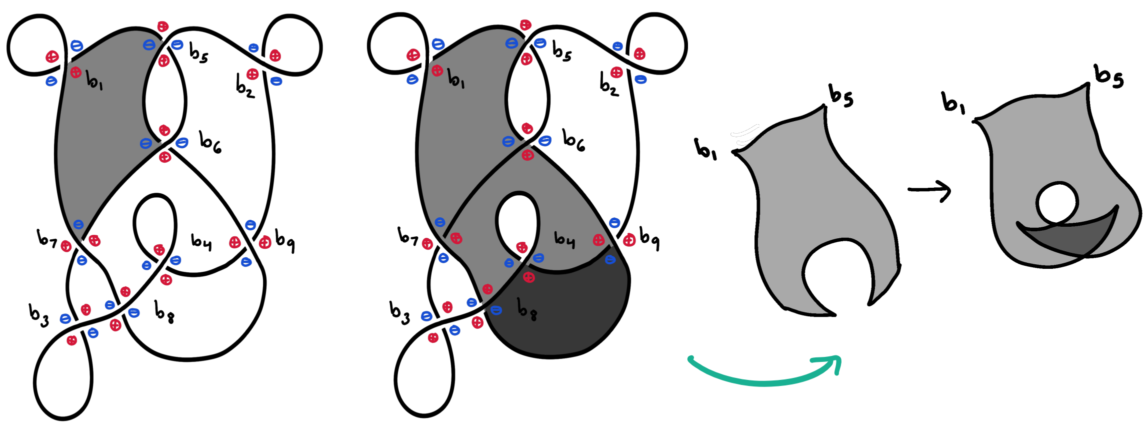
1 $\quad$ a_7 $\quad$ $a_7 a_6 a_5$

$$\partial_A(a_1) = 1 + a_7 + a_7 a_6 a_5$$

(B, ∂_B)



1 $\quad$ b_7 $\quad$ b_5



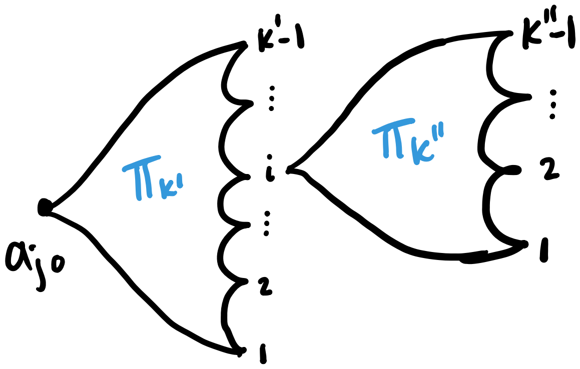
$b_7 b_6 b_5$ $\quad$ $b_9 b_8 b_5$

$$\partial_B(b_1) = 1 + b_7 + b_5 + b_7 b_6 b_5 + b_9 b_8 b_5$$

III $\partial^2 = 0$

"combinatorial imitation of gluing-compactness argument used by Floer to prove $\partial^2=0$ in the Lagrangian intersection complex"

- suffices to show $\partial^2(a_i)=0$ $\forall$ crossings a_i
- heuristic (geometry not accurate)



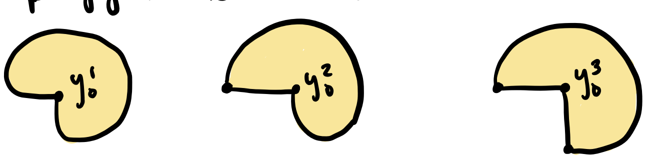
the k-gon w/ vertices as in (c)
$(k'+k''=k+2)$

- $N :=$ # crossings
- $\partial^2(a_{j_0}) = \sum_{k\geq 0} \sum_{1\leq j_1,\dots,j_{k-1}\leq N} \boxed{\left|\mathcal{U}(\Pi(L), a_{j_0}, \dots, a_{j_{k-1}})\right|}\, a_{j_1}, \dots, a_{j_{k-1}}$

cardinality of the set $\mathcal{U}(-)$ consisting of triples (f', i, f'') st

(a) $f' \in W_{k'}(\Pi(L), a_{j_0})$
(b) $f'' \in W_{k''}(\Pi(L), f'(x_i^{k'}))$ and $i \in \{1, \dots, k'-1\}$
(c) $(a_{j_1}, \dots, a_{j_{k-1}}) = (f'(x_1^{k'}), \dots, f'(x_{i-1}^{k'}), f''(x_1^{k''}), \dots, f''(x_{k''-1}^{k''}), f'(x_{i+1}^{k'}), \dots, f'(x_{k'-1}^{k'}))$

- goal: all $|\mathcal{U}(-)|$ are even ($\equiv 0 \mod 2$)
- $\Theta_k :=$ polygon $\subseteq \mathbb{R}^2$ w/ k vertices and one angle larger than π (at y_0^k)



Θ_1 Θ_2 Θ_3 …

each of these will correspond to pairs of two immersions

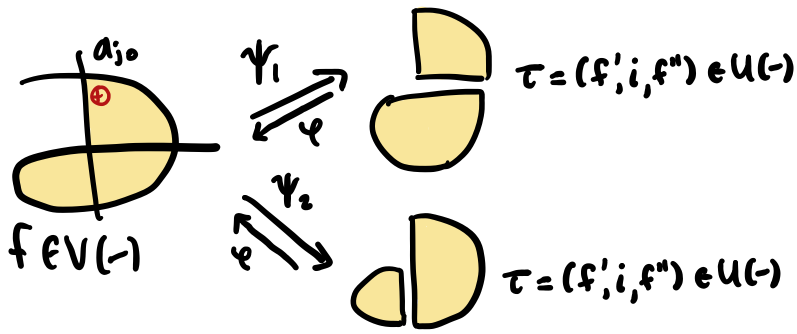
= "admissible immersions" of θ_K's (only the vertex sent to a_{j_0} is pos.)

- **tools**: $V(\pi(L), a_{j_0}, \dots, a_{j_{k-1}})$ auxillary space of immersions

  $\varphi: U(-) \longrightarrow V(-)$ gluing map

  $\psi_1, \psi_2: V(-) \longrightarrow U(-)$ cutting maps

e.g.



such that

- $\psi_1(f) \neq \psi_2(f) \quad \forall f \in V(-)$

  cutting maps cut the same immersion differently

- $f = \varphi(\psi_1(f)) = \varphi(\psi_2(f)) \quad \forall f \in V(-)$

  cutting two diff. ways then gluing back together

- $\tau = \psi_1(\varphi(\tau))$ XOR $\tau = \psi_2(\varphi(\tau)) \quad \forall \tau \in U(-)$

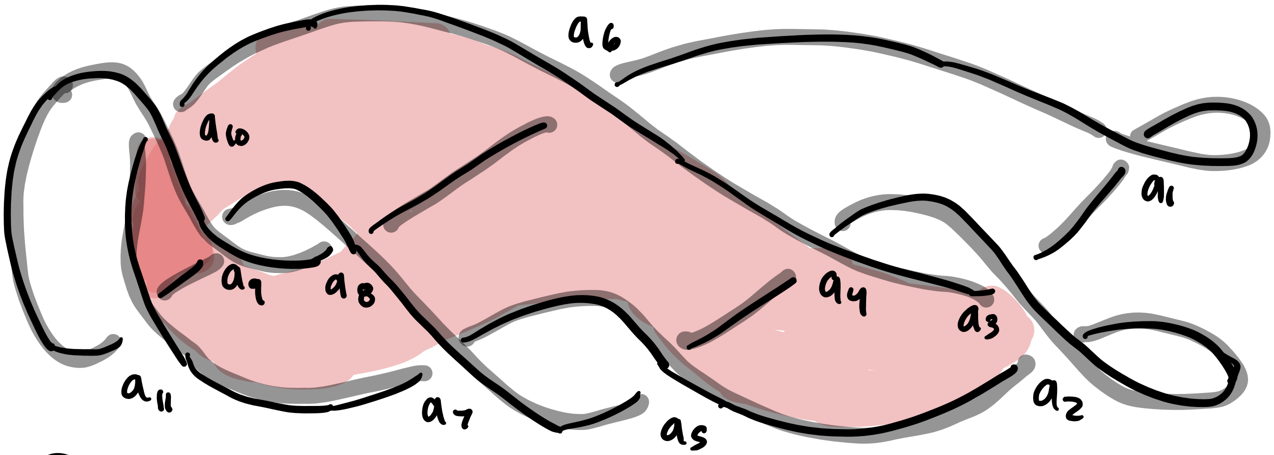
  after gluing two polygons to get one in $V(-)$, can only cut one way to get back to original polygons

- **done**: this implies $|U(-)|$ is even, because the cardinality of $U(-)$ is twice that of $V(-)$

Ex looking at $\partial^2(a_2)$

$\partial(a_6) = a_{11}a_8$
$\partial(a_7) = a_8 a_{10}$
$\partial(a_3) = 0$
$\partial(a_{10}) = 0$
$\partial(a_{11}) = 0$

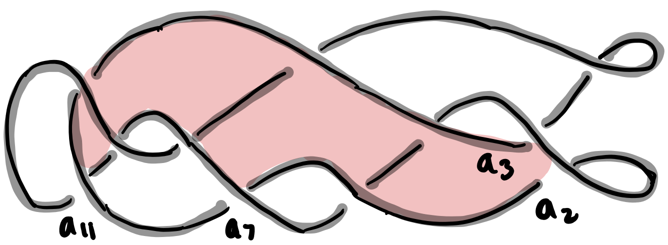
$\partial(a_2)$ includes $a_3a_{11}a_7$ and $a_3a_6a_{10}$,
so $\partial^2(a_2)$ includes $a_3a_{11}\partial(a_7)$ and $a_3\partial(a_6)a_{10}$



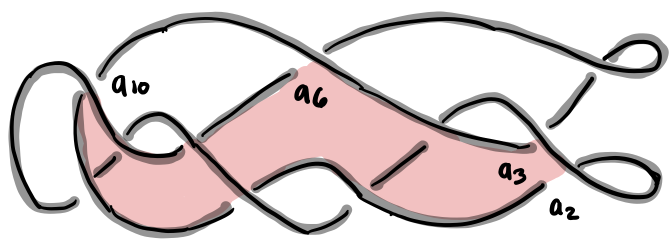
$$\mathcal{O}_K = a_2 a_3 a_{11} a_8 a_{10} = a_3 a_{11}\partial(a_7) = a_3\partial(a_6)a_{10}$$

$\swarrow \psi_1$ $\qquad\qquad$ $\searrow \psi_2$

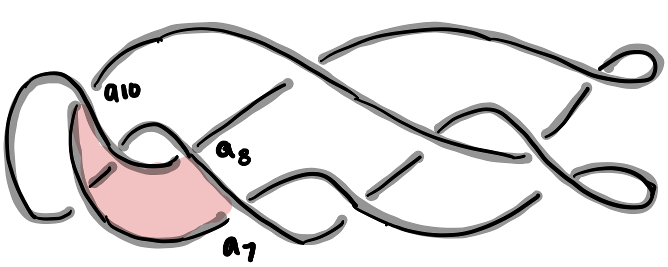
Π_K': $a_2a_3a_{11}a_7$



Π_K' : $a_2a_3a_6a_{10}$



Π_K'': $a_7a_8a_{10}$



Π_K'' : $a_6a_{11}a_8$