Sarah Blackwell

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# Lagrangian Cobordisms between Legendrians

trisector jr. skype mtg notes

**[Recall]**

contact manifold $(Y^{2n+1}, \xi)$
legendrian submanifold $\Lambda^n : T\Lambda \subset \xi$
symplectic manifold $(X^{2n+2}, \omega)$
Lagrangian submanifold $L^{n+1} : \omega|_{TL} \equiv 0$

Symplectic mfld $(M^{2n}, \omega)$ (closed 2-form w/ non-deg. condition) $d\omega = 0$ $\omega^n \neq 0$ (Darboux thm) locally $\omega = dx_1 \wedge dy_1 + \dots + dx_n \wedge dy_n$ on $\mathbb{R}^{2n}$

$T^*X$, $d\lambda_{can}$ symp. str.

every function $H$ on $(M, \omega)$ generates a flow (this stuff came out of physics)

Contact geometry on $Y^{2n+1}$, $\xi$ 2n-dim (hyperplane) field, maximally non integrable (more of a geometric structure/interpretation)

if $L \subseteq Y$ submfld, $TL \subseteq \xi \Rightarrow \dim L \leq n$
Legendrian if =

$\xi = \ker\alpha$ $\quad \alpha \wedge (d\alpha)^n \neq 0$ (a similar Darboux thm)

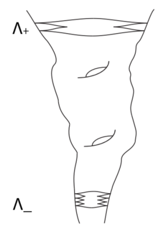
$\mathbb{R}^{2n+1}$ $\alpha = dz + \sum x_i dy_i$ (std)

**[Setting]**

contact manifold $\to$ $\mathbb{R}^3$ $\quad \alpha = dz + x\,dy$ $\quad$ (not canonical ... $\pi/2$ rot)

symplectization $\to$ $\mathbb{R}^4_+$ $\quad \mathrm{Symp}(\mathbb{R}^3, \ker\alpha) = (\mathbb{R}_{>0} \times \mathbb{R}^3, d(e^t\alpha))$ $(t)$

- a Lagrangian cobordism between Legendrian $\Lambda_\pm$, denoted $L_{\Lambda_-}^{\Lambda_+}$, consists of a properly embedded Lagrangian submanifold of $(\mathbb{R}_{\geq 0} \times \mathbb{R}^3, d(e^t\alpha))$ st $\exists\, s_0, t_0 \in \mathbb{R}$ st $L \cap (\mathbb{R}^3 \times \{t\}) = \Lambda_- \times \{t\}$ $\forall t < s_0$ and $L \cap (\mathbb{R}^3 \times \{t\}) = \Lambda_+ \times \{t\}$ $\forall t > t_0$
- when $\Lambda_- = \emptyset$, $L_{\emptyset}^{\Lambda_+}$ is a Lagrangian filling of $\Lambda_+$
- when $\Lambda_+ = \emptyset$, $L_{\Lambda_-}^{\emptyset}$ is a Lagrangian cap of $\Lambda_-$




## — Classical Obstructions —

- theorem [Chantraine]: If $\Lambda_\pm$ are Legendrian knots in $\mathbb{R}^3$ (can be generalized to a contact 3-mfld), and $L_{\Lambda_-}^{\Lambda_+}$ is an orientable Lagrangian cobordism of genus $g$ between them, then

$$r(\Lambda_-) = r(\Lambda_+)$$
$$tb(\Lambda_+) - tb(\Lambda_-) = -\chi(L) = 2g - 1$$

If $\Lambda_\pm$ admits a Lagrangian cap/filling, then $r(\Lambda_\pm) = 0$.

For a filling of $\Lambda_+$

$$tb(\Lambda_+) = -\chi(L) = 2g - 1$$

For a cap of $\Lambda_-$

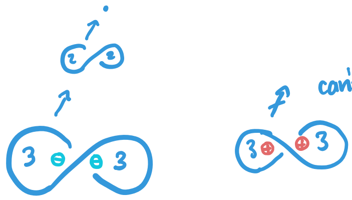
$$tb(\Lambda_-) = \chi(L) = 1 - 2g$$

# Lin-Murphy Moves

$R_0$:


Can't pick A st pos area changes to neg or viceversa

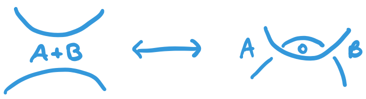
ex




can't shrink, but could grow

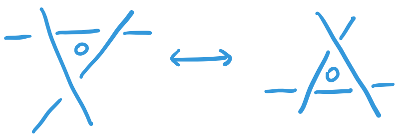
("0" is never really 0)

$R_2$:



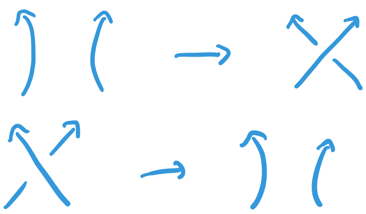

if A or B not 0

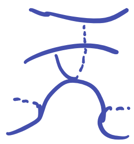
$R_3$:




$H_+$:

$H_-$:



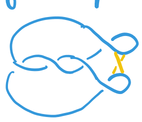


handle attachment

"very happy surgery"
create pos crossings
destroy neg crossings

prop: $K_-$ has L.D. $D_-$, $K_+$ has L.D. commensurate to $D_+$ (not a big change)
$D_\pm$ are related by one of $R_0, R_2, R_3, H_\pm$ moves.
$\therefore K_\pm$ can be related as an embedded Lag. cob $L$ in $\mathbb{R}_3 \times [0,\varepsilon]$ in the sense that
$\Sigma \cap \mathbb{R}_3 \times \{0\} = K_-$
$\Sigma \cap \mathbb{R}_3 \times \{\varepsilon\} = K_+$

prop: $\Sigma$ is a surface built by $R_0, R_2, R_3, H_\pm$ together with capping, $\Sigma$ is exact if
- at each slice, total area is 0 (for every component of the link has area 0)
- if you do a handle attachment ($H_+$ or $H_-$) merging two components into one component, the xy-projs of the components are disjoint



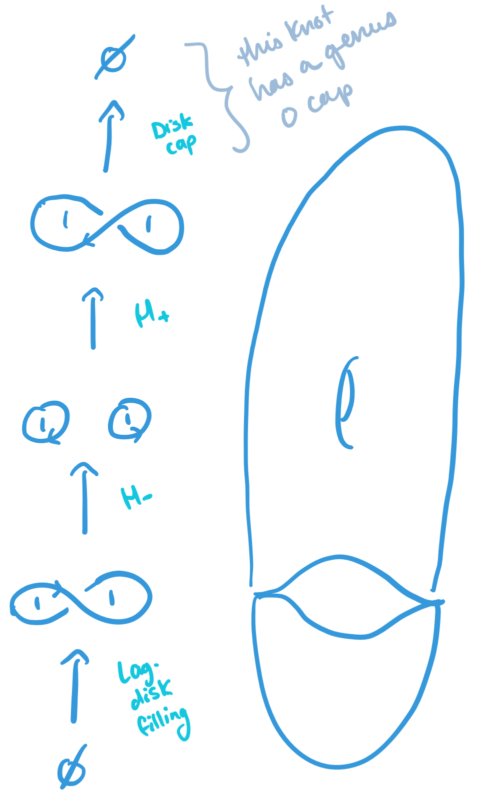
not (definitely) exact

sometimes care about exact Lag. cob.
($e^s \alpha|_L = df$ where $f =$ constant$_\pm$ outside $[s_-, s_+]$)

ex

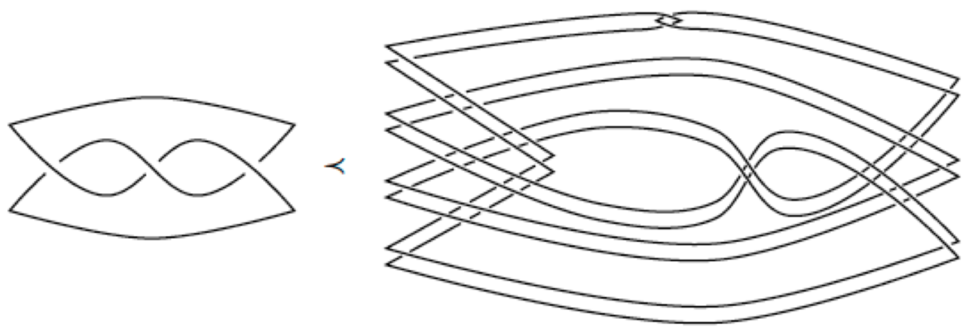
this knot has a genus 0 cap

not total area 0




can't be exact

Fact:



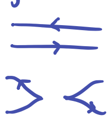
conj: ∄ decomposable cob.

higher up in symplectization ($t$)

$\Lambda_- = \Lambda_0 < \Lambda_1 < \Lambda_2 < \cdots < \Lambda_n = \Lambda_+$

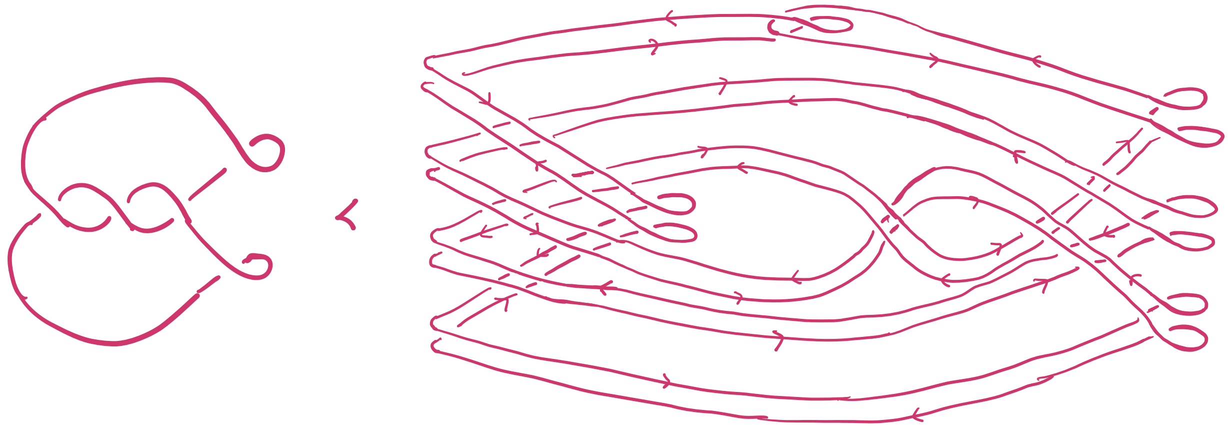
where $\Lambda_{i+1}$ obtained from $\Lambda_i$ by one of:

(1) Legendrian isotopy

(2)  handle attachment

(3)  $\Lambda_i$ / $\Lambda_i$ add local min.

goal: use Lin-Murphy moves to construct cob.



idea