Differential Graded Algebras of Legendrian Knots

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Mock AMS

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Goal: an invariant of Legendrian knots that can distinguish Legendrian knots with the same "classical invariants"

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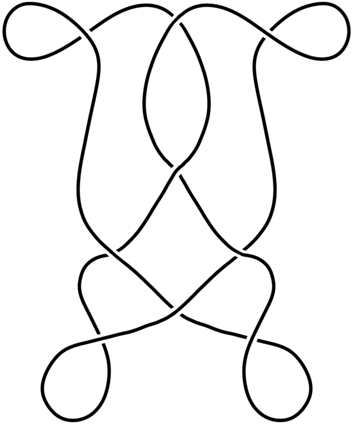
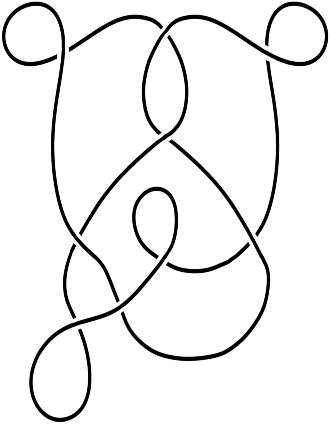
Chekanov (1997): invariant using DGAs

Legendrian Knots Background

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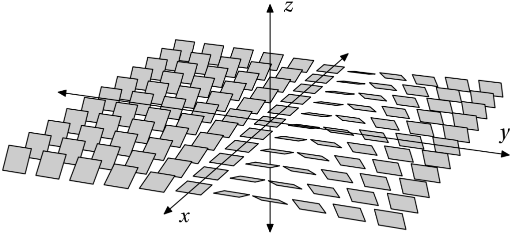

wikipedia.org/wiki/Contact_geometry

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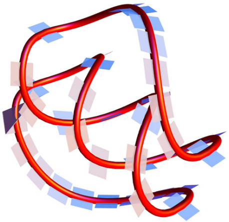

services.math.duke.edu/~ng/knotgallery.html

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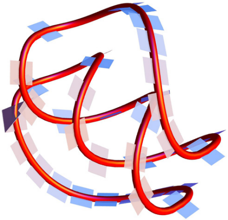

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- Two Legendrian knots are **Legendrian isotopic** if they can be connected by a smooth 1-parameter family of Legendrian knots

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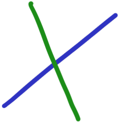

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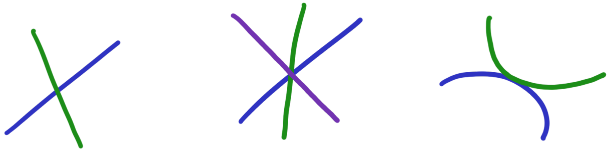

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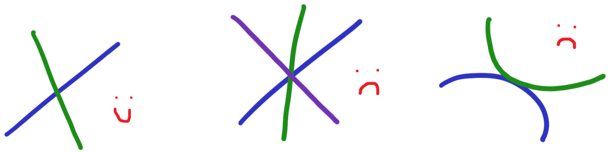

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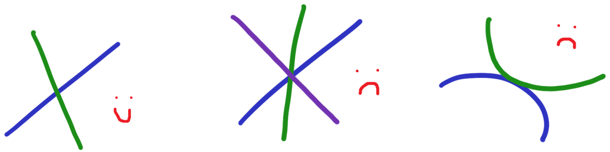

- The **diagram** of a π-generic Legendrian knot L is $\pi(L)$

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Classical Invariants

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1. **Smooth isotopy type**

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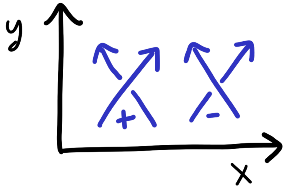

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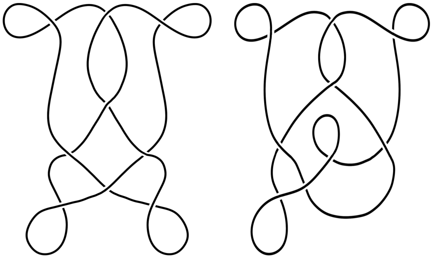
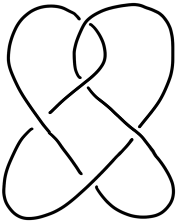


Smooth isotopy type $= 5_2$



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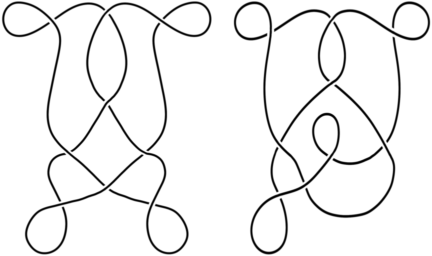

Maslov number $m(L) =$ twice the rotation number of $\pi(L) = 0$

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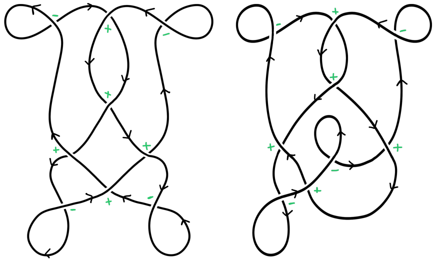
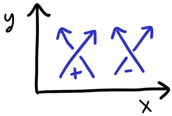


Thurston-Bennequin number $\beta(L)$ = signed count of the crossings of $\pi(L) = 1$



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$$(A, \partial)$$

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- AKA the algebra of non-commutative polynomials in variables $a_1, \ldots, a_n$ and with coefficients in $\mathbb{Z}_2$
- Define $\deg(a_i)$ so that $(T(a_1, \ldots, a_n), \partial)$ becomes a DGA $\rightsquigarrow$ **semi-free DGA**

Associating a DGA to a Legendrian Knot

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   - $\deg(a) \in \mathbb{Z}/m(L)\mathbb{Z}$

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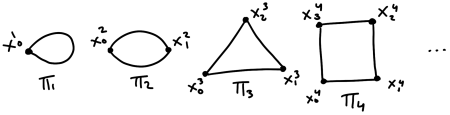
- Π_k = (curved) convex k-gon with vertices $x_0^k, \ldots, x_{k-1}^k$ numbered counter-clockwise



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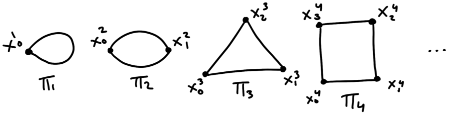

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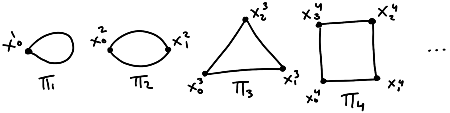

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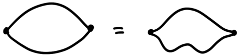
- Furthermore: consider classes of immersions that fix vertices



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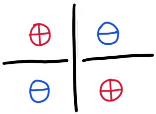

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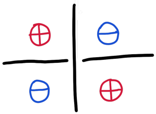

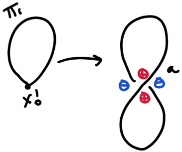

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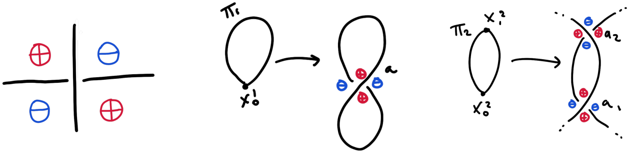

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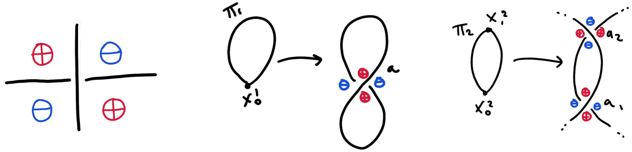

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- $\partial = \sum_{k \geq 0} \partial_k$ where

$$\partial_k(a) = \sum_{f \in W_{k+1}(\pi(L), a)} f(x_1^{k+1}) \cdots f(x_k^{k+1})$$

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- Extend ∂ to A by linearity and Leibniz rule

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Theorem (Chekanov): $\partial^2 = 0$

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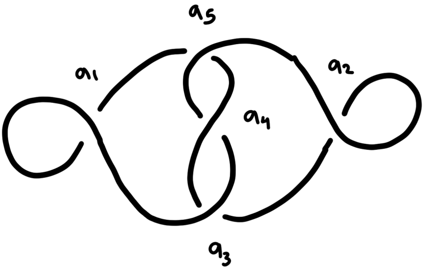

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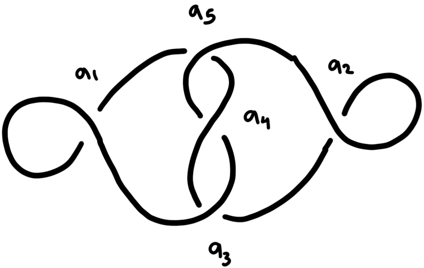

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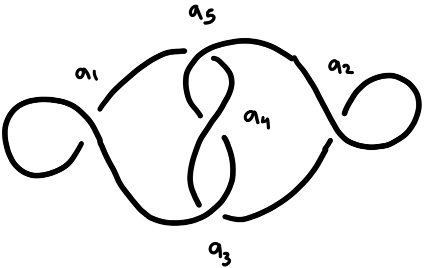

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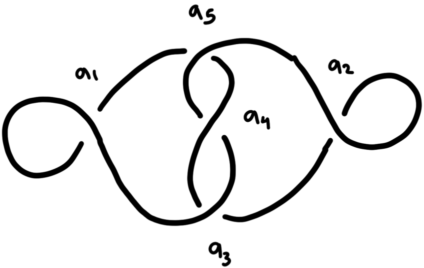

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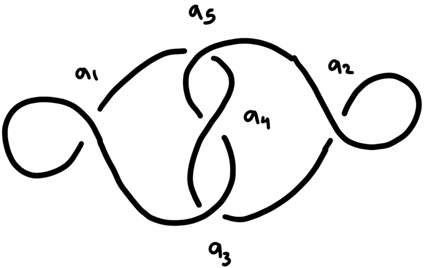

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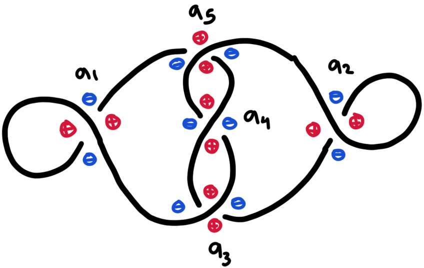

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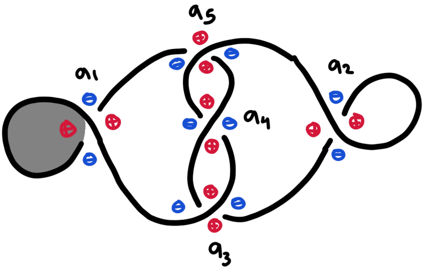

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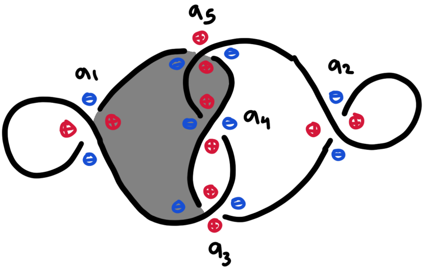

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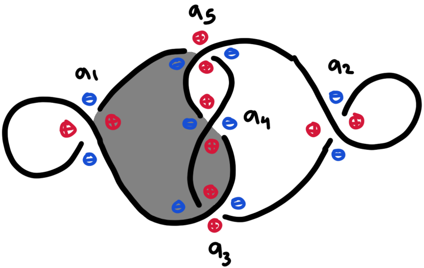

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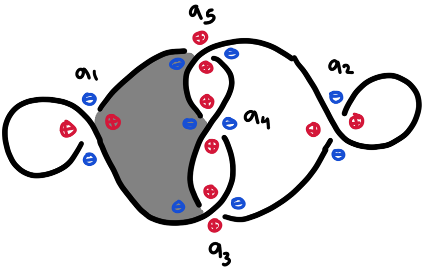

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- $\partial(a_1) = 1 + a_3 + a_5 + a_3 a_4 a_5$

Associating a DGA to a Legendrian Knot

Theorem (Chekanov): Let $(A, \partial), (A', \partial')$ be the DGAs of π-generic Legendrian knots L, L'. If L is Legendrian isotopic to L' then $(A, \partial), (A', \partial')$ *have the same stable type*.

The 5_2 Example

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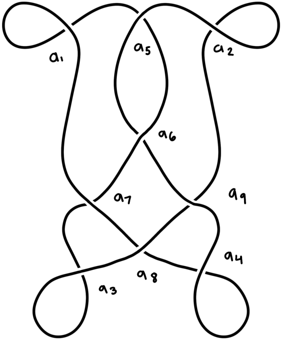

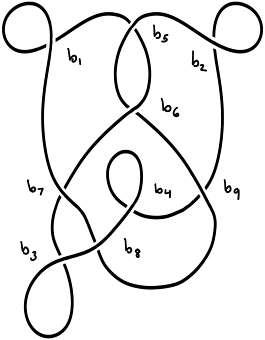

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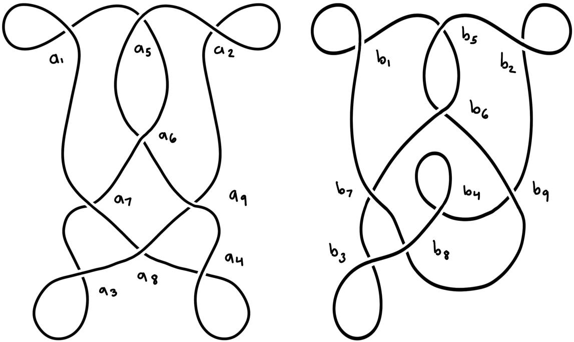

$$\partial(a_1) = ?$$

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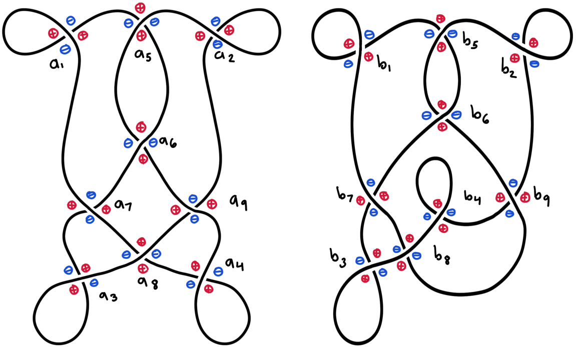

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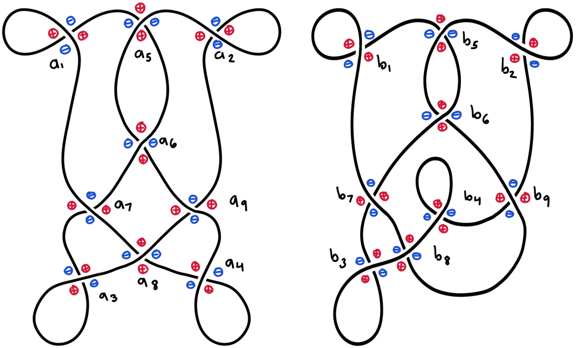

$$\partial(a_1) = 1 + a_7 + a_7 a_6 a_5$$

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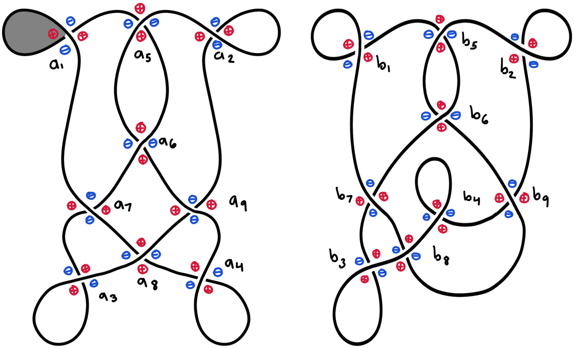

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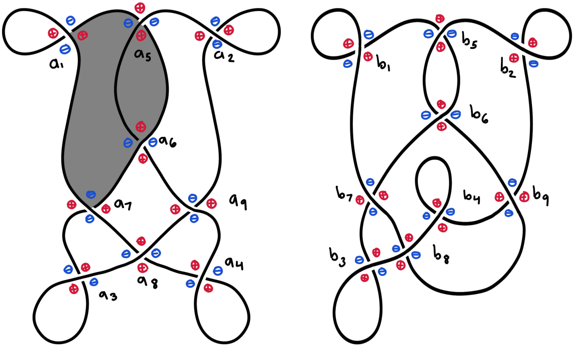

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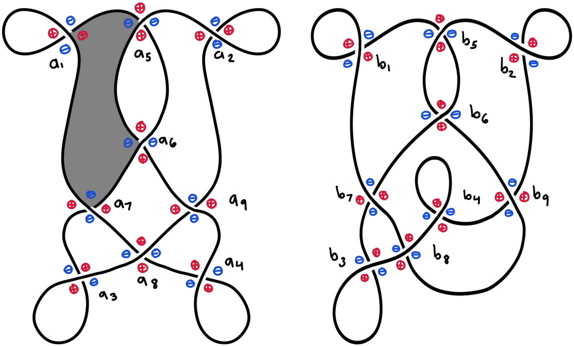

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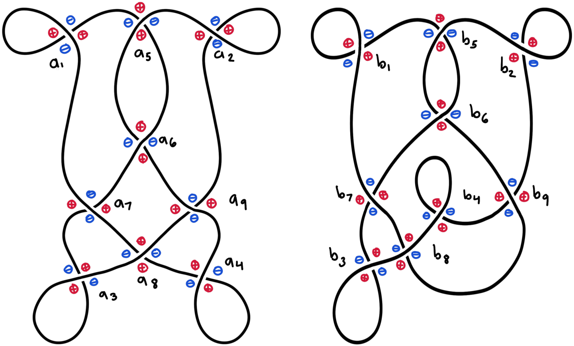

$$\partial(a_1) = 1 + a_7 + a_7 a_6 a_5$$

$$\partial(b_1) = 1 + b_7 + b_5 + b_7 b_6 b_5 + b_9 b_8 b_5$$

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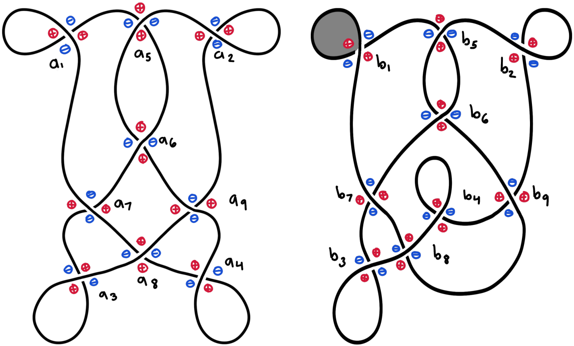

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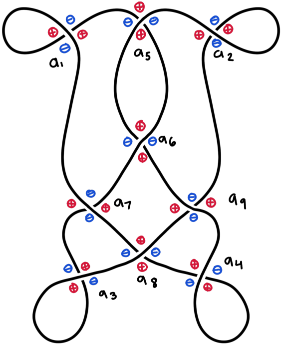

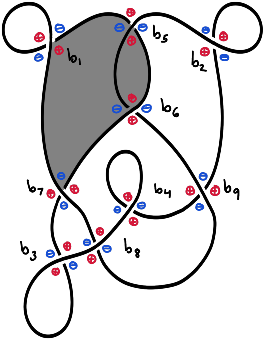

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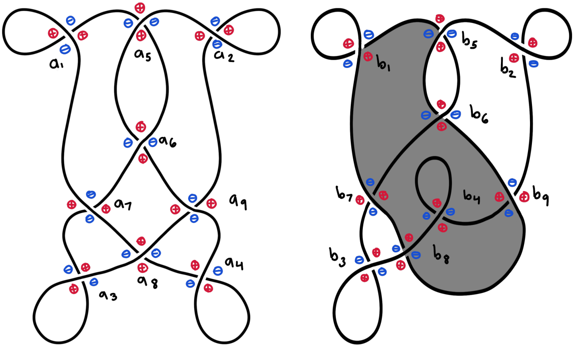

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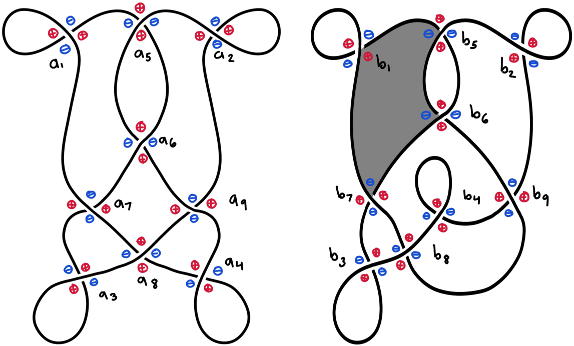

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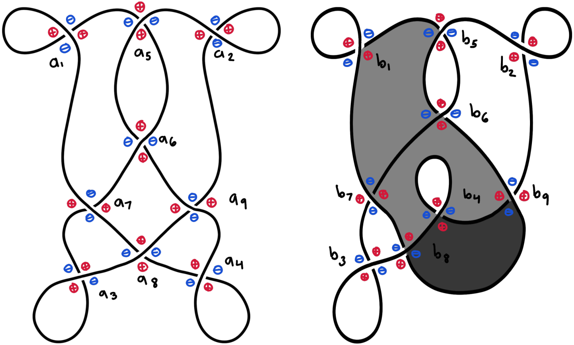

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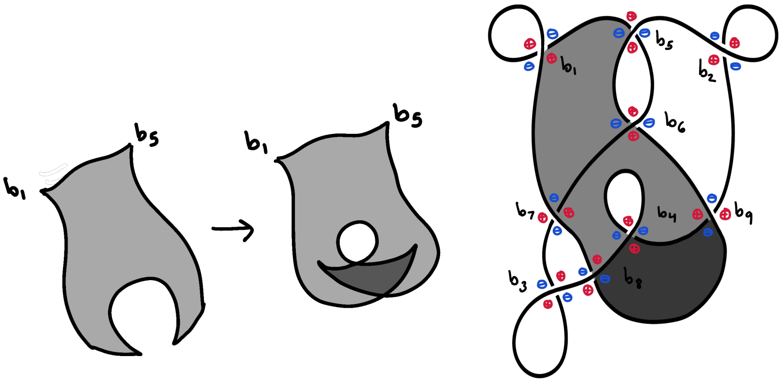

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How do we know these DGAs are different?

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How do we know these DGAs are different?

Need to use *stable type invariants*...

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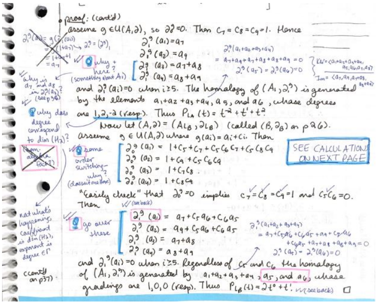

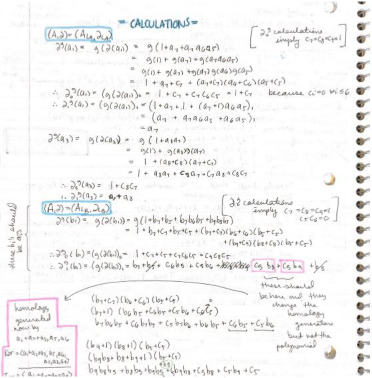

Thank you!