GROUP TRISECTIONS + SMOOTHLY KNOTTED SURFACES

Sarah Blackwell
University of Georgia

j.w. Kirby, Klug, Longo, Ruppik

* everything is smooth! *

BIG PICTURE

topology		algebra
trisected 4-mfld X	$\xrightarrow{\text{apply van Kampen thm to pieces}}$	group trisection of $\pi_1(X)$
	$\xleftarrow[\text{abrams, Gay, Kirby}]{}$	

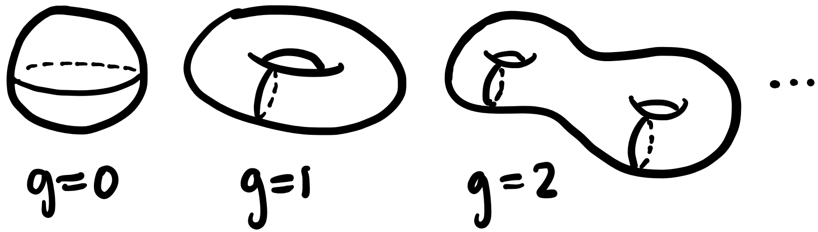
* everything is smooth! *

BIG PICTURE

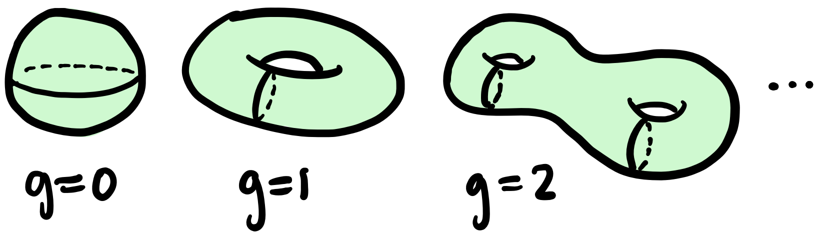
topology — **algebra**

$$\begin{tikzcd}[column sep=huge]
\text{trisected 4-mfld } X \arrow[r, shift left, "\text{apply van Kampen thm to pieces}"] & \text{group trisection of } \pi_1(X) \arrow[l, shift left, "\text{abrams, Gay, Kirby}"] \\
\text{bridge trisected surface } S \arrow[u, "\text{inherited from}"'] \arrow[r, shift left, "\text{apply van Kampen thm to pieces}"] & \text{group trisection of } \pi_1(X \setminus S) \arrow[l, shift left, "\text{B.K.K.L.R.}"]
\end{tikzcd}$$

Σ_g = genus g surface



H_g = genus g handlebody



TRISECTIONS

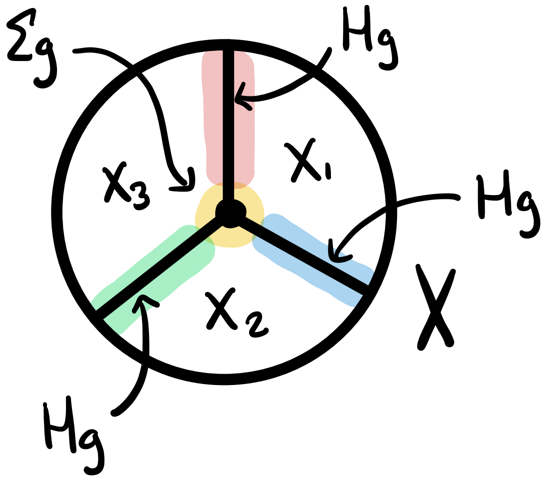
$\underline{(g,k)\text{-trisection}}$ of smooth, closed, connected, oriented 4-mfld X = decomposition $X = X_1 \cup X_2 \cup X_3$ st

[Gay + Kirby]

1) $X_i \cong \natural^k (S^1 \times B^3)$

2) $X_i \cap X_j \cong H_g$ = genus g handlebody

3) $X_1 \cap X_2 \cap X_3 \cong \Sigma_g$ = genus g surface



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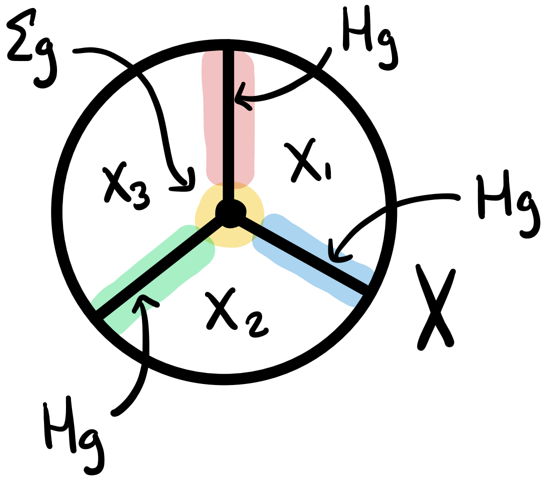
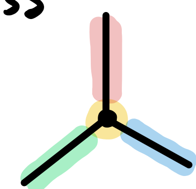
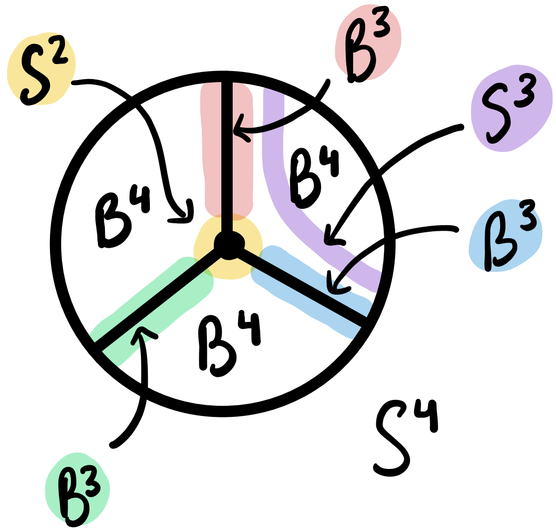


- existence + uniqueness
- determined by "spine"



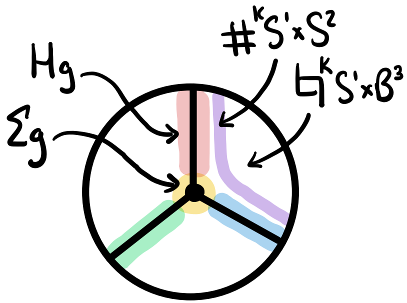
TRISECTIONS

ex (unique) genus 0 trisection of S^4



GROUP TRISECTIONS

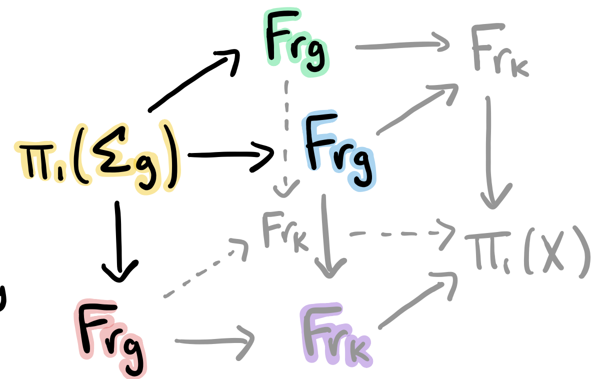
<u>(g, K)-group trisection</u> of a group G = commutative cube of groups (as shown below) st every homomorphism is surjective and each face is a pushout



(g, K)- trisection of 4-mfld X

van Kampen thm $\longrightarrow$

$\longleftarrow$ Abrams, Gay, Kirby



(g, K)- group trisection of $\pi_1(X)$

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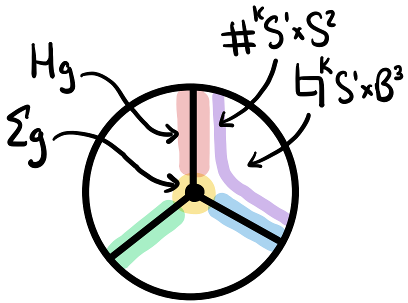


(g,k)-trisection of 4-mfld X

$\xrightarrow{\text{van Kampen thm}}$

$\xleftarrow{\text{Abrams, Gay, Kirby}}$

$$
\begin{tikzcd}
& F_{r_g} \arrow[rr] \arrow[dd, dashed] & & F_{r_k} \arrow[dd] \\
\pi_1(\Sigma_g) \arrow[ur] \arrow[rr] \arrow[dd] & & F_{r_g} \arrow[ur] \arrow[dd] & \\
& F_{r_k} \arrow[rr, dashed] & & \pi_1(X) \\
F_{r_g} \arrow[rr] \arrow[ur, dashed] & & F_{r_k} \arrow[ur] &
\end{tikzcd}
$$

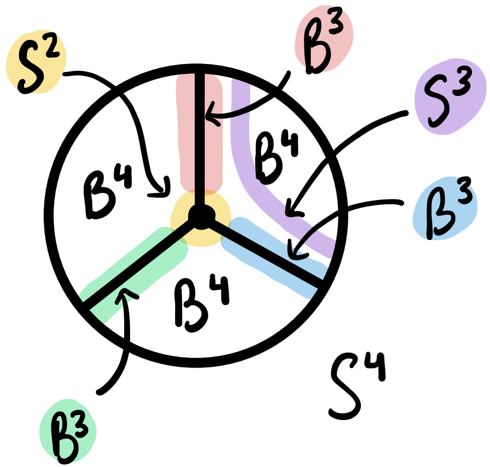
(g,k)-group trisection of $\pi_1(X)$

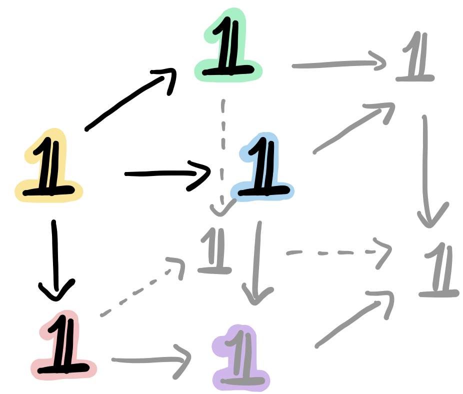
- existence + uniqueness
- determined by "tripod"

$$
\begin{tikzcd}
& F_{r_g} \\
\pi_1(\Sigma_g) \arrow[ur] \arrow[r] \arrow[d] & F_{r_g} \\
F_{r_g} &
\end{tikzcd}
$$

GROUP TRISECTIONS

ⓔⓧ group trisection of $\mathbb{1}$ coming from genus 0 trisection of S^4





BRIDGE TRISECTIONS

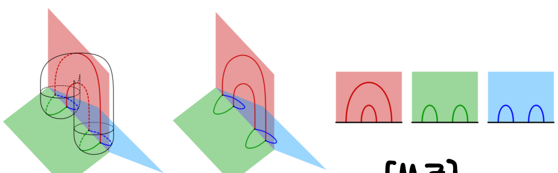
<u>(b,c)-bridge trisection</u> of a smoothly knotted surface S in S^4 = decomposition
$(S^4, S) = (X_1, D_1) \cup (X_2, D_2) \cup (X_3, D_3)$ st

[Meier + Zupan]

1) $S^4 = X_1 \cup X_2 \cup X_3$ is the genus 0 trisection of S^4

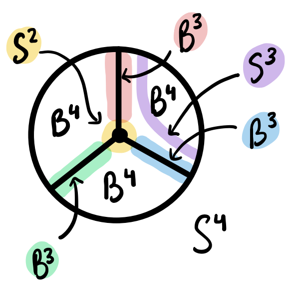
2) (X_i, D_i) is a trivial disk system: $X_i = B^4$ and $D \subset B^4$ a collection of c properly embedded disks which are simultaneously isotopic into ∂B^4

3) $(B_{ij}, \alpha_{ij}) = (X_i, D_i) \cap (X_j, D_j)$ is a b-strand trivial tangle: $B_{ij} = B^3$ and $\alpha_{ij} \subset B^3$ a collection of b properly embedded B' which are ambiently isotopic into ∂B^3 relative $\partial \alpha$



[M-Z]

FIGURE 4. A 2–bridge trisection of an unknotted 2–sphere, depicted with the tri-plane in 3–space, along with the corresponding tri-plane diagram.



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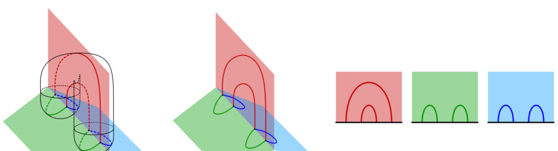

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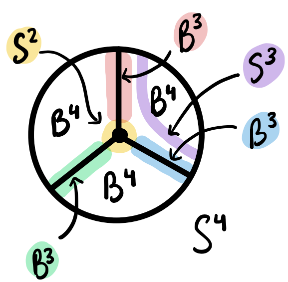
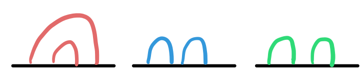
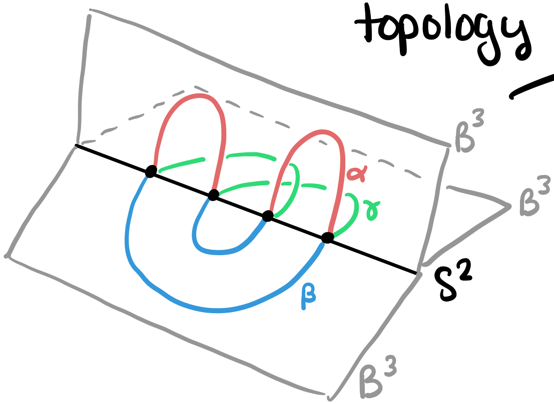


- existence + uniqueness
- determined by tangles 

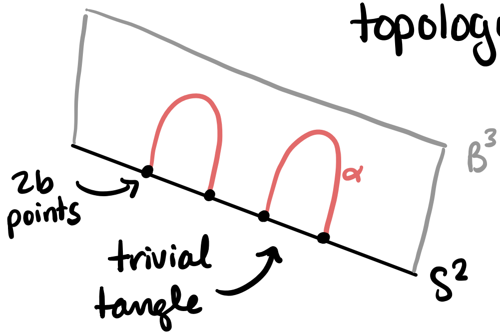
PUTTING IT ALL TOGETHER...



topology $\longrightarrow$ algebra

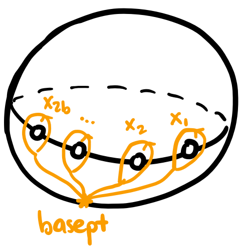
$$\begin{tikzcd}
 & \pi_1(B^3_3\setminus\gamma) \arrow[rr, two heads] \arrow[dd, dashed, two heads] & & \pi_1(S^3_3\setminus\mathcal{U}_{\beta\gamma}) \arrow[dd, two heads] \\
\pi_1(S^2\setminus 2b\,*) \arrow[ur, two heads] \arrow[rr, two heads] \arrow[dd, two heads] & & \pi_1(B^3_2\setminus\beta) \arrow[ur, two heads] \arrow[dd, two heads] & \\
 & \pi_1(S^3_2\setminus\mathcal{U}_{\gamma\alpha}) \arrow[rr, dashed, two heads] & & \pi_1(S^4\setminus S) \\
\pi_1(B^3_1\setminus\alpha) \arrow[rr, two heads] \arrow[ur, dashed, two heads] & & \pi_1(S^3_1\setminus\mathcal{U}_{\alpha\beta}) \arrow[ur, two heads] & \\
 & & \cong \pi_1(B^4_1\setminus D_{\alpha\beta}) &
\end{tikzcd}$$

PUTTING IT ALL TOGETHER...



topology $\longrightarrow$ algebra

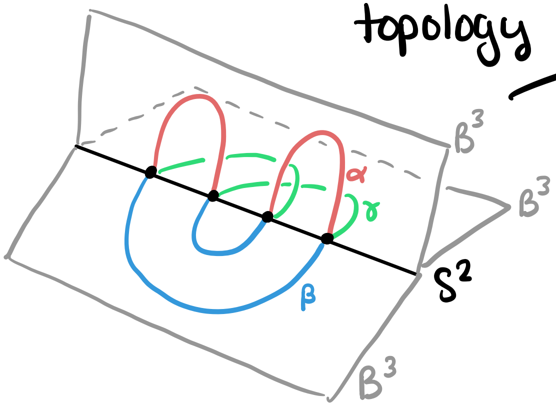
$$\pi_1(S^2 \setminus 2b *) \to \pi_1(B^3_i \setminus \alpha)$$



$$\therefore \ \pi_1(S^2 \setminus 2b *) \cong \underset{\displaystyle Fr_{2b-1}}{\overset{\cong}{\langle \bar{x}_1, \dots, \bar{x}_{2b-1} \rangle}} \cong \underset{\displaystyle Fr_{2b}/\sim}{\overset{\cong}{\langle x_1, \dots, x_{2b} \rangle}} \Big/ {x_1 \cdots x_{2b} = 1}$$

$$\pi_1(B^3 \setminus \alpha) \cong Fr_b$$

PUTTING IT ALL TOGETHER...



topology

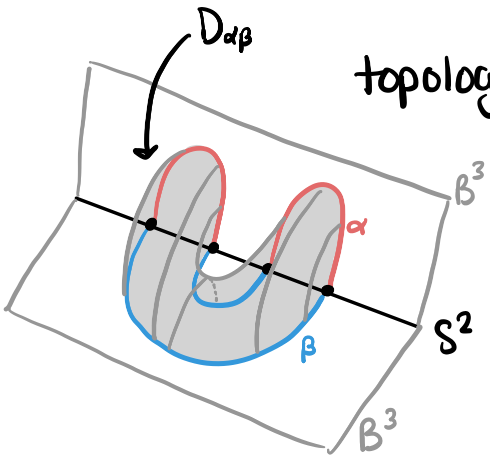
algebra

$$\begin{tikzcd} & \pi_1(B^3_3\setminus\gamma) \\ \pi_1(S^2\setminus 2b*) \arrow[ur, two heads] \arrow[r, two heads] \arrow[d, two heads] & \pi_1(B^3_2\setminus\beta) \\ \pi_1(B^3_1\setminus\alpha) & \end{tikzcd}$$

at this point the rest of the cube is determined!
(Keep pushing out)

but let's talk about it anyway

PUTTING IT ALL TOGETHER...



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\pi_1(B^3_1 \setminus \alpha) \arrow[rr, two heads] & & \pi_1(S^3_1 \setminus \mathcal{U}_{\alpha\beta}) \\
 & & \cong \pi_1(B^4_1 \setminus D_{\alpha\beta})
\end{tikzcd}$$

$\mathcal{U}_{\alpha\beta} = \alpha \cup \beta$ (unlink)

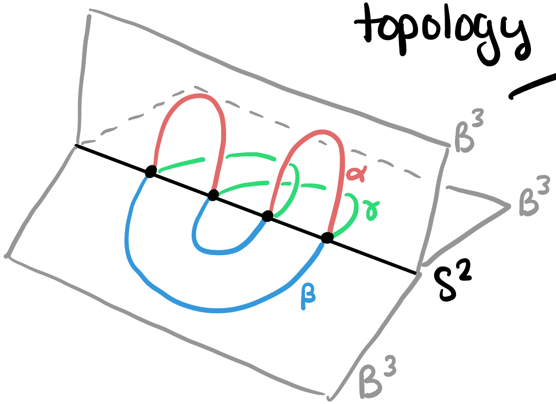
$$\pi_1(S^3_1 \setminus \mathcal{U}_{\alpha\beta}) \cong \pi_1(B^4_1 \setminus D_{\alpha\beta}) \cong F_{r_c}$$

unlink
$\partial D_{\alpha\beta} = \mathcal{U}_{\alpha\beta}$

$\bigsqcup_c D^2$

"patch #"
= # disks needed to cap off $\mathcal{U}_{\alpha\beta}$

PUTTING IT ALL TOGETHER...



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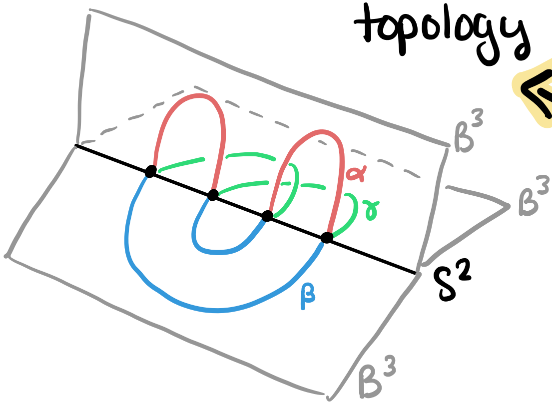
 & \pi_1(S^3_2 \setminus \mathcal{U}_{\gamma\alpha}) \arrow[rr, dashed] & & \pi_1(S^4 \setminus S) \\
\pi_1(B^3_1 \setminus \alpha) \arrow[ur, dashed] \arrow[rr, two heads] & & \pi_1(S^3_1 \setminus \mathcal{U}_{\alpha\beta}) \arrow[ur] &
\end{tikzcd}$$

$\cong \pi_1(B^4_1 \setminus D_{\alpha\beta})$

repeat w/ all three tangles to finish building the surface S

$$\begin{tikzcd}
 & Fr_b \arrow[rr] \arrow[dd, dashed] & & Fr_{c_2} \arrow[dd] \\
Fr_{2b}/rel \arrow[ur] \arrow[rr] \arrow[dd] & & Fr_b \arrow[ur] \arrow[dd] & \\
 & Fr_{c_3} \arrow[rr, dashed] & & \pi_1(S^4 \setminus S) \\
Fr_b \arrow[ur, dashed] \arrow[rr] & & Fr_{c_1} \arrow[ur] &
\end{tikzcd}$$

THEOREM [B.K.K.L.R.]:



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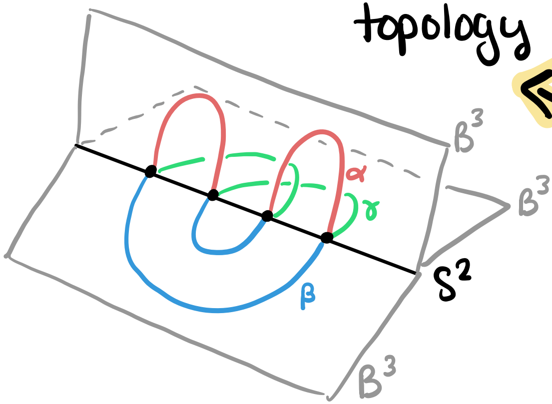

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\end{tikzcd}
$$

$\cong \pi_1(B^4_1\setminus D_{\alpha\beta})$

- more precise statement in forthcoming paper
- have extended to surfaces in general 4-manifolds
- proof involves an algorithm + Stallings folding

$\therefore$ π_1 of knotted surface complement in S^4 does not determine the knotted surface, but the trisection of π_1 does determine it