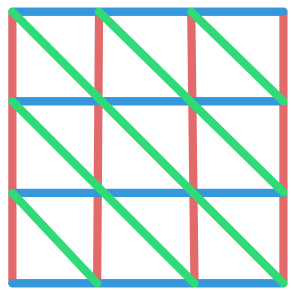
TRIPLE KNOT GRID DIAGRAMS

Sarah Blackwell
University of Georgia

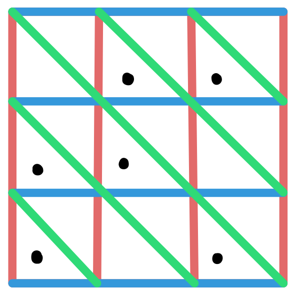
DEFINITION

a **triple knot grid diagram** (TKGD) is a diagram on a torus made up of vertical, horizontal, and diagonal grid lines like so–



DEFINITION

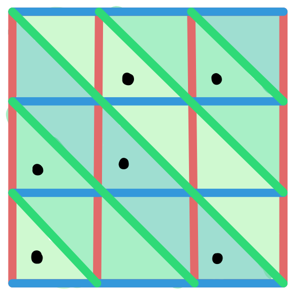
a **triple knot grid diagram** (TKGD) is a diagram on a torus made up of vertical, horizontal, and diagonal grid lines like so –



– with dots placed inside st each row, column, and diagonal has exactly two or zero dots

DEFINITION

a triple knot grid diagram (TKGD) is a diagram on a torus made up of vertical, horizontal, and diagonal grid lines like so -

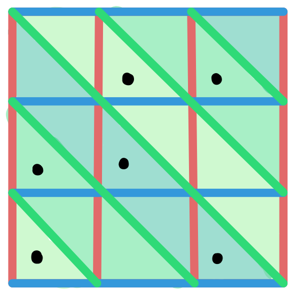


(use lower triangles)

- with dots placed inside st each row, column, and diagonal has exactly two or zero dots

DEFINITION

a triple knot grid diagram (TKGD) is a diagram on a torus made up of vertical, horizontal, and diagonal grid lines like so –

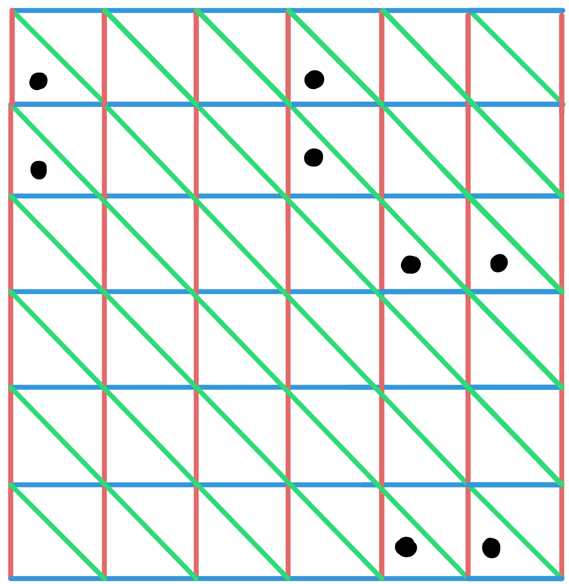


(use lower triangles)

- with dots placed inside st each row, column, and diagonal has exactly two or zero dots

DEFINITION

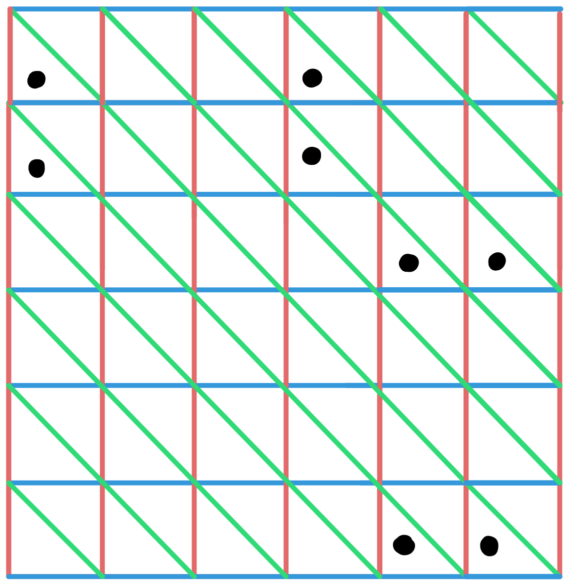
- with dots placed inside st each row, column, and diagonal has exactly two or zero dots



DEFINITION

- with dots placed inside st each row, column, and diagonal has exactly two or zero dots

the grid number for a TKGD is n if the grid is nxn



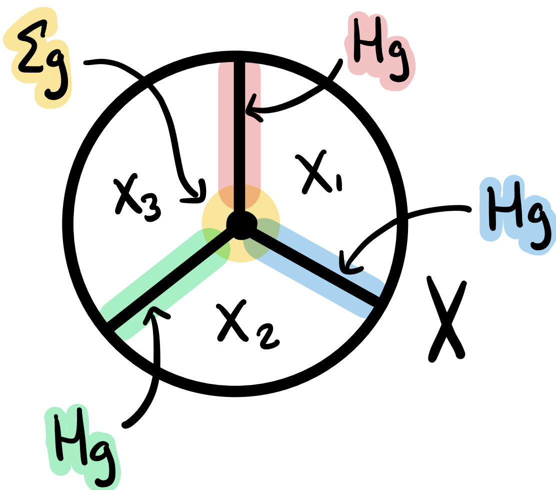
"You're probably
wondering how
I got here..."

TRISECTIONS

$\underline{(g,k)\text{-trisection}}$ of smooth, closed, connected, oriented 4-mfld X = decomposition $X = X_1 \cup X_2 \cup X_3$ st

[Gay + Kirby]

1) $X_i \cong \natural^k (S^1 \times B^3)$ = 4D 1-handlebody
2) $X_i \cap X_j \cong H_g$ = genus g handlebody
3) $X_1 \cap X_2 \cap X_3 \cong \Sigma_g$ = genus g surface

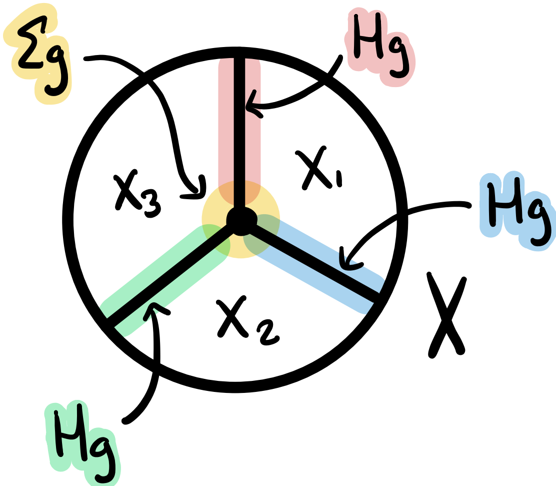


TRISECTIONS

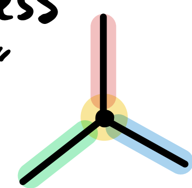
(g,k)-trisection of smooth, closed, connected, oriented 4-mfld X = decomposition $X = X_1 \cup X_2 \cup X_3$ st

[Gay + Kirby]

1) $X_i \cong \natural^k (S^1 \times B^3)$ = 4D 1-handlebody
2) $X_i \cap X_j \cong H_g$ = genus g handlebody
3) $X_1 \cap X_2 \cap X_3 \cong \Sigma_g$ = genus g surface



- existence + uniqueness
- determined by "spine"

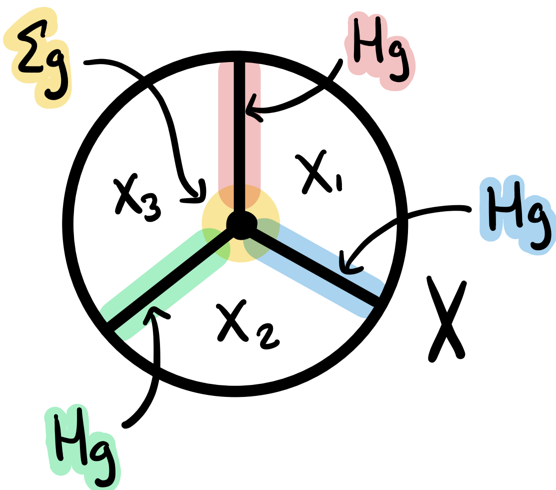


TRISECTIONS

(g,k)-trisection of smooth, closed, connected, oriented 4-mfld X = decomposition $X = X_1 \cup X_2 \cup X_3$ st

[Gay + Kirby]

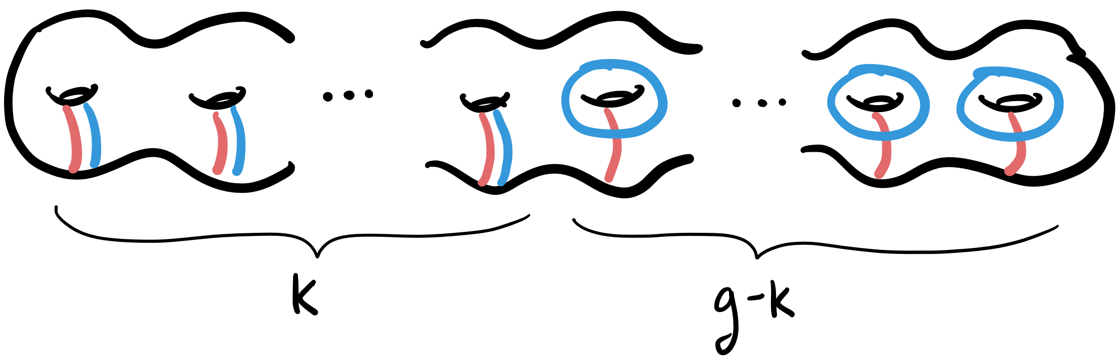
1) $X_i \cong \natural^k (S^1 \times B^3)$ = 4D 1-handlebody
2) $X_i \cap X_j \cong H_g$ = genus g handlebody
3) $X_1 \cap X_2 \cap X_3 \cong \Sigma_g$ = genus g surface



- 4D analogue of Heegaard splittings
- $\partial X_i = H_g \cup H_g$ is a Heegaard splitting for $\partial X_i \cong \partial(\natural^k(S^1 \times B^3)) \cong \#^k(S^1 \times S^2)$

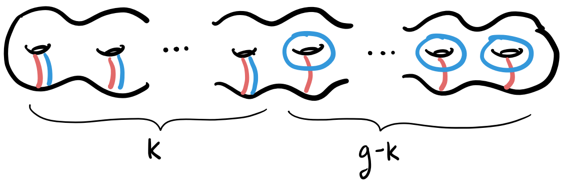
TRISECTIONS

<u>(g,k)- trisection diagram</u> $= (\Sigma_g; \alpha, \beta, \gamma)$ s.t. $(\Sigma_g; \alpha, \beta)$, $(\Sigma_g; \beta, \gamma)$, $(\Sigma_g; \gamma, \alpha)$ are equiv. to the standard Heegaard diagram for $\#^k(S^1 \times S^2)$

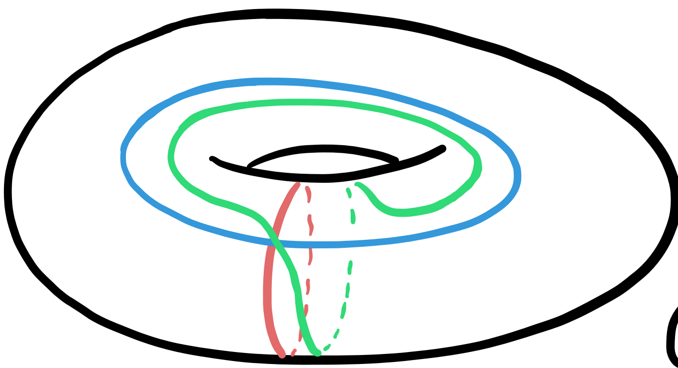


TRISECTIONS

(g,k)-trisection diagram $= (\Sigma_g;\alpha,\beta,\gamma)$ s.t. $(\Sigma_g;\alpha,\beta)$, $(\Sigma_g;\beta,\gamma)$, $(\Sigma_g;\gamma,\alpha)$ are equiv. to the standard Heegaard diagram for $\#^k(S^1\times S^2)$

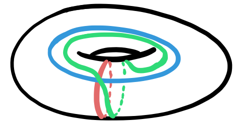


ex

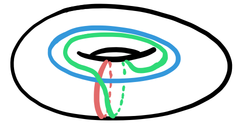


$\mathbb{CP}^2$

TRISECTIONS

ex $\mathbb{CP}^2$ admits a $(1,0)$-trisection 

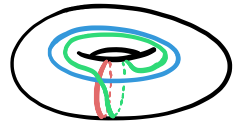
TRISECTIONS

ex $\mathbb{CP}^2$ admits a (1,0)-trisection 

$$\mathbb{CP}^2 = \{[z_1 : z_2 : z_3] : z_i \in \mathbb{C},\ [z_1 : z_2 : z_3] = [\lambda z_1 : \lambda z_2 : \lambda z_3],\ \lambda \in \mathbb{C}\setminus\{0\}\}$$

TRISECTIONS

ex $\mathbb{CP}^2$ admits a (1,0)-trisection

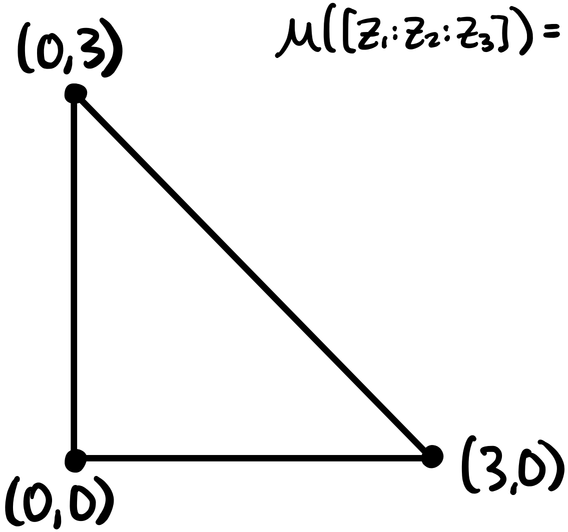


$$\mathbb{CP}^2 = \{[z_1 : z_2 : z_3] : z_i \in \mathbb{C},\ [z_1 : z_2 : z_3] = [\lambda z_1 : \lambda z_2 : \lambda z_3],\ \lambda \in \mathbb{C}\setminus\{0\}\}$$

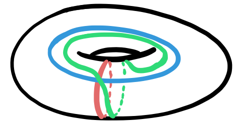
"moment map" $\mu : \mathbb{CP}^2 \to \mathbb{R}^2$

$$\mu([z_1 : z_2 : z_3]) = \left(\frac{3|z_1|^2}{|z_1|^2 + |z_2|^2 + |z_3|^2},\ \frac{3|z_2|^2}{|z_1|^2 + |z_2|^2 + |z_3|^2} \right)$$

image of μ



TRISECTIONS

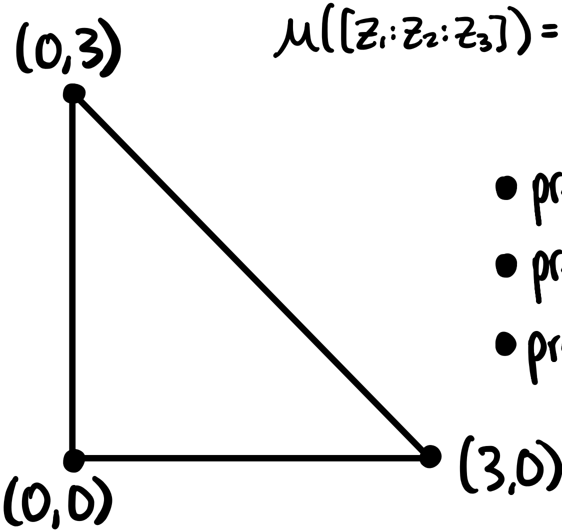
ex $\mathbb{CP}^2$ admits a (1,0)-trisection 

$$\mathbb{CP}^2 = \{[z_1:z_2:z_3] : z_i \in \mathbb{C},\ [z_1:z_2:z_3] = [\lambda z_1 : \lambda z_2 : \lambda z_3],\ \lambda \in \mathbb{C}\setminus\{0\}\}$$

"moment map" $\mu : \mathbb{CP}^2 \to \mathbb{R}^2$

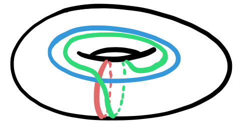
$$\mu([z_1:z_2:z_3]) = \left(\frac{3|z_1|^2}{|z_1|^2+|z_2|^2+|z_3|^2},\ \frac{3|z_2|^2}{|z_1|^2+|z_2|^2+|z_3|^2} \right)$$

image of μ



- preimage of corner pts = pts
- preimage of pts inside edges = circles
- preimage of interior pt = torus

TRISECTIONS

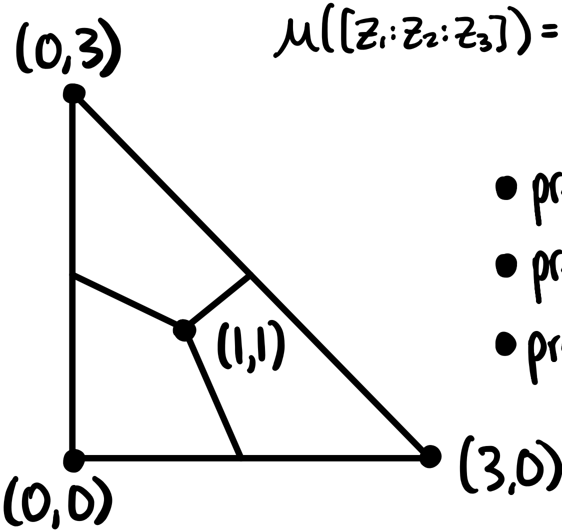
ex $\mathbb{CP}^2$ admits a $(1,0)$-trisection 

$$\mathbb{CP}^2 = \{[z_1:z_2:z_3] : z_i \in \mathbb{C},\ [z_1:z_2:z_3] = [\lambda z_1 : \lambda z_2 : \lambda z_3],\ \lambda \in \mathbb{C}\setminus\{0\}\}$$

"moment map" $\mu : \mathbb{CP}^2 \to \mathbb{R}^2$

$$\mu([z_1:z_2:z_3]) = \left(\frac{3|z_1|^2}{|z_1|^2+|z_2|^2+|z_3|^2},\ \frac{3|z_2|^2}{|z_1|^2+|z_2|^2+|z_3|^2} \right)$$

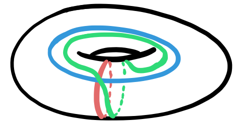
image of μ



- preimage of corner pts = pts
- preimage of pts inside edges = circles
- preimage of interior pt = torus

TRISECTIONS

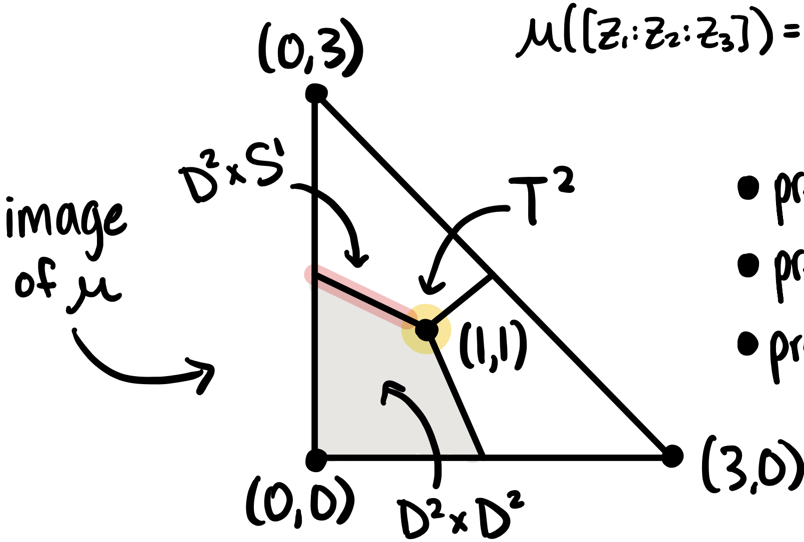
ex $\mathbb{CP}^2$ admits a (1,0)-trisection



$$\mathbb{CP}^2 = \{[z_1:z_2:z_3] : z_i \in \mathbb{C},\ [z_1:z_2:z_3] = [\lambda z_1 : \lambda z_2 : \lambda z_3],\ \lambda \in \mathbb{C}\setminus\{0\}\}$$

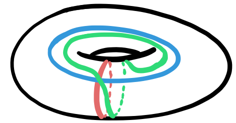
"moment map" $\mu : \mathbb{CP}^2 \to \mathbb{R}^2$

$$\mu([z_1:z_2:z_3]) = \left(\frac{3|z_1|^2}{|z_1|^2+|z_2|^2+|z_3|^2},\ \frac{3|z_2|^2}{|z_1|^2+|z_2|^2+|z_3|^2} \right)$$



- preimage of corner pts = pts
- preimage of pts inside edges = circles
- preimage of interior pt = torus

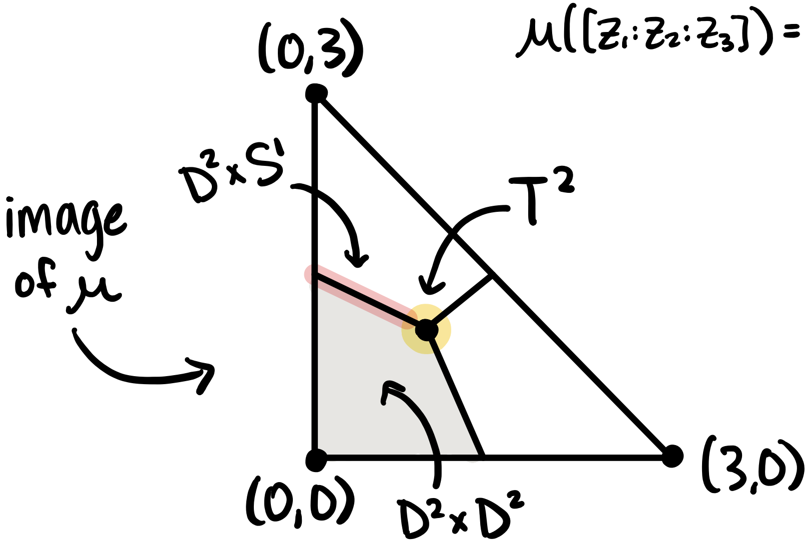
TRISECTIONS

ex $\mathbb{CP}^2$ admits a $(1,0)$-trisection 

$$\mathbb{CP}^2 = \{[z_1:z_2:z_3] : z_i \in \mathbb{C},\ [z_1:z_2:z_3] = [\lambda z_1:\lambda z_2:\lambda z_3],\ \lambda \in \mathbb{C}\setminus\{0\}\}$$

"moment map" $\mu : \mathbb{CP}^2 \to \mathbb{R}^2$

$$\mu([z_1:z_2:z_3]) = \left(\frac{3|z_1|^2}{|z_1|^2+|z_2|^2+|z_3|^2},\ \frac{3|z_2|^2}{|z_1|^2+|z_2|^2+|z_3|^2} \right)$$



$$\Sigma_g = T^2$$

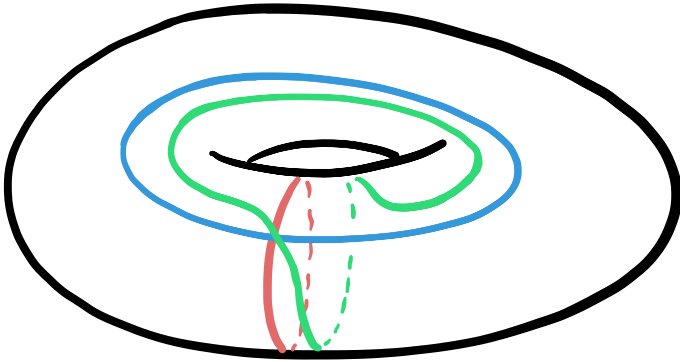
$$H_g = D^2 \times S^1$$

$$X_i = B^4$$

now back to
the grids!

SETUP: (1,0)-TRISECTION OF $\mathbb{CP}^2$

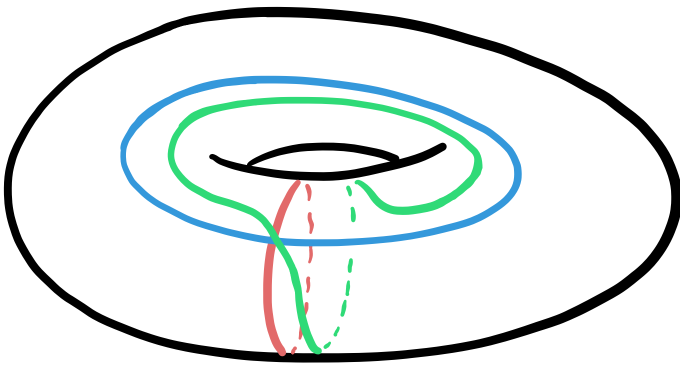
decomposition into three 4-balls
("0")



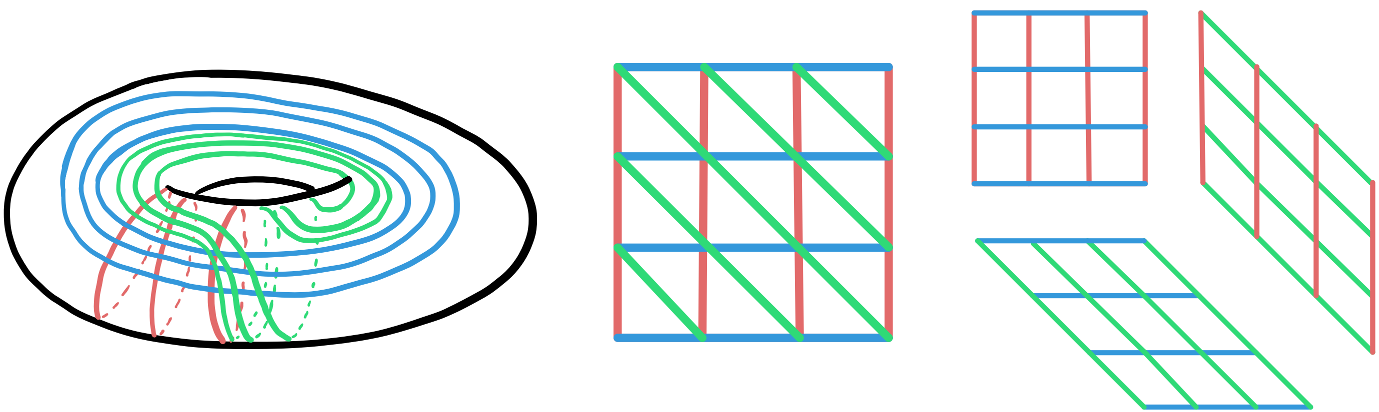
triple intersection of 4-balls
("1")

SETUP: (1,0)-TRISECTION OF $\mathbb{CP}^2$

decomposition into three 4-balls
("0")

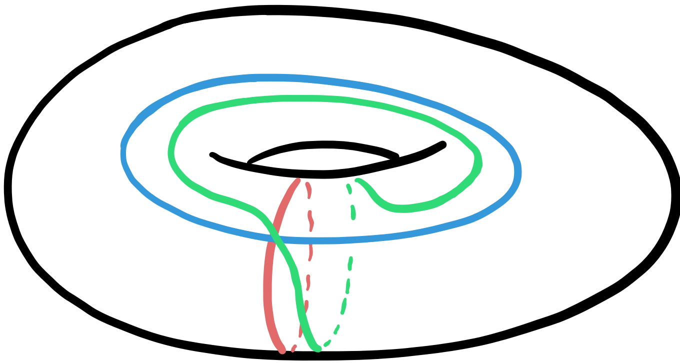


triple intersection of 4-balls
("1")

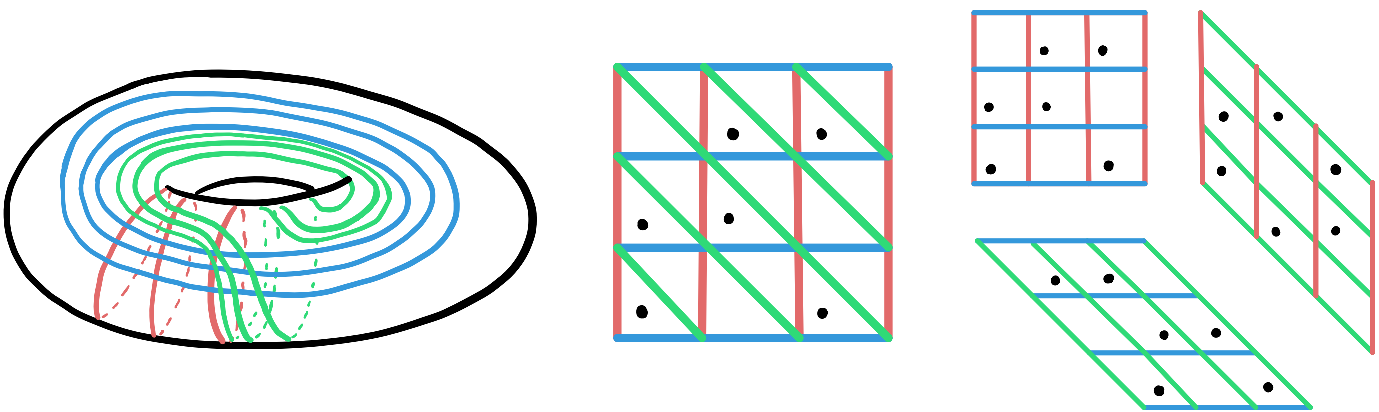


SETUP: (1,0)-TRISECTION OF $\mathbb{CP}^2$

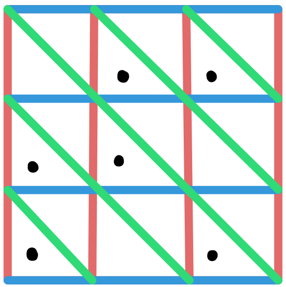
decomposition into three 4-balls
("0")

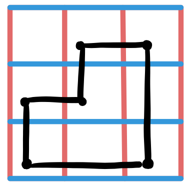


triple intersection of 4-balls
("1")

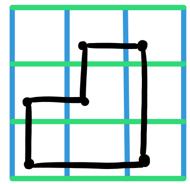


SURFACES IN $\mathbb{CP}^2$

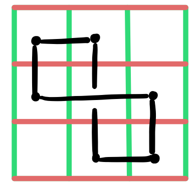




blue over red

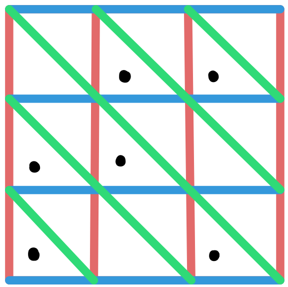


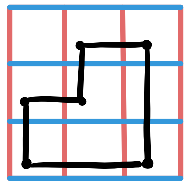
green over blue



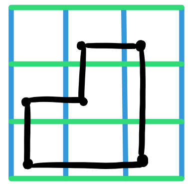
red over green

SURFACES IN $\mathbb{CP}^2$

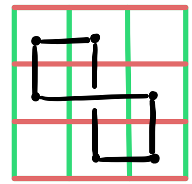




blue over red



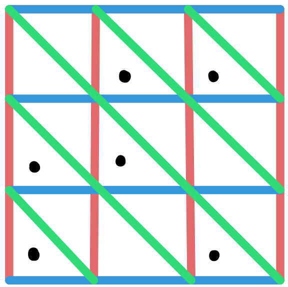
green over blue

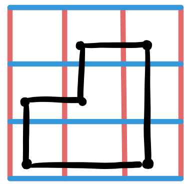


red over green

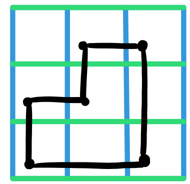
if all three links are unlinks, can cap off w/ disks $\rightsquigarrow$ embedded surface in $\mathbb{CP}^2$

SURFACES IN $\mathbb{CP}^2$

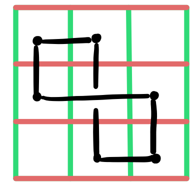




blue over red



green over blue

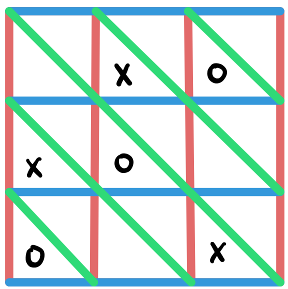


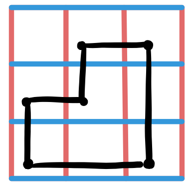
red over green

if all three links are unlinks, can cap off w/ disks $\rightsquigarrow$ embedded surface in $\mathbb{CP}^2$

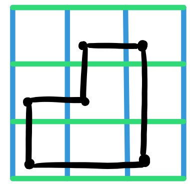
(TKGD is a particular shadow diagram for the bridge-trisected surface)
[Meier-Zupan]

SURFACES IN $\mathbb{CP}^2$

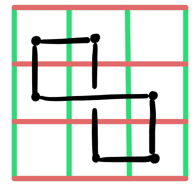




blue over red



green over blue



red over green

if all three links are unlinks, can cap off w/ disks $\rightsquigarrow$ embedded surface in $\mathbb{CP}^2$

a TKGD is orientable if X's and O's can be placed consistantly, and orientable TKGD $\rightsquigarrow$ orientable surface

GEOMETRY INTERLUDE

GEOMETRY INTERLUDE

maximally non-integrable hyperplane field

contact manifold (Y^{2n+1}, ξ)

Legendrian submanifold Λ^n: $T\Lambda \subset \xi$

GEOMETRY INTERLUDE

maximally non-integrable hyperplane field

contact manifold (Y^{2n+1}, ξ)

Legendrian submanifold Λ^n: $T\Lambda \subset \xi$

closed, non-degenerate 2-form

symplectic manifold (X^{2n+2}, ω)

Lagrangian submanifold L^{n+1}: $\omega|_L \equiv 0$

GEOMETRY INTERLUDE

maximally non-integrable hyperplane field

contact manifold (Y^3, ξ)

Legendrian submanifold $\Lambda^1 : T\Lambda \subset \xi$

closed, non-degenerate 2-form

symplectic manifold (X^4, ω)

Lagrangian submanifold $L^2 : \omega|_L \equiv 0$

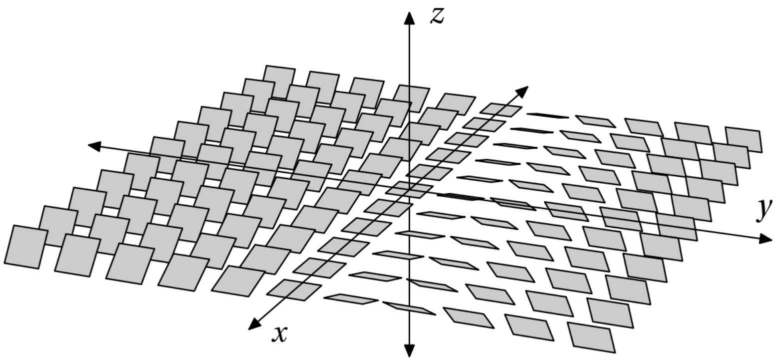
GEOMETRY INTERLUDE

maximally non-integrable hyperplane field

contact manifold (Y^3, ξ)

Legendrian link

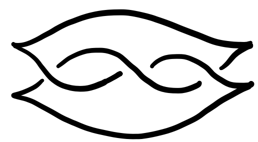
Legendrian submanifold $\Lambda^1 : T\Lambda \subset \xi$



front projection of Λ:

$\pi : \mathbb{R}^3 \to \mathbb{R}^2$

$(x,y,z) \mapsto (x,z)$



right-handed trefoil

z, x

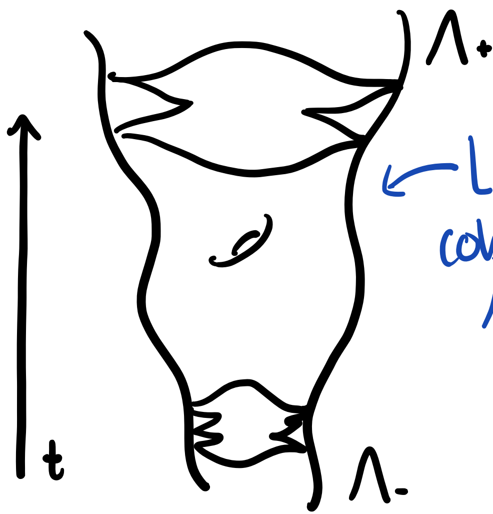
"Legendrian front"

GEOMETRY INTERLUDE

closed, non-degenerate 2-form

symplectic manifold (X^4, ω)

Lagrangian submanifold L^2 : $\omega|_L \equiv 0$

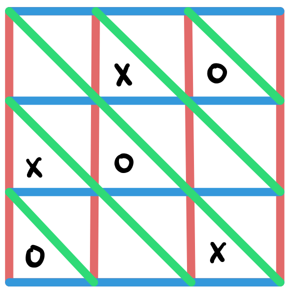


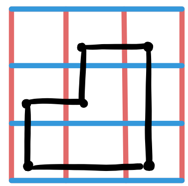
Lagrangian cobordism btw Λ_- and Λ_+

filling

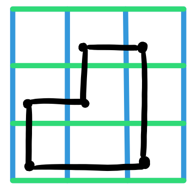
cap

SURFACES IN $\mathbb{CP}^2$

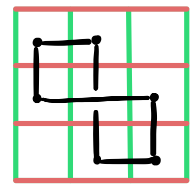




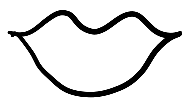
blue over red

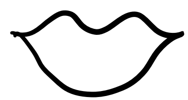


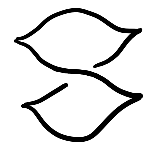
green over blue



red over green

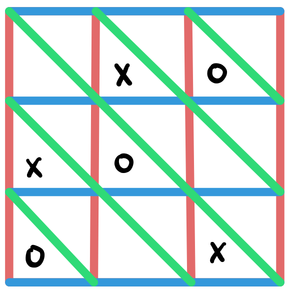


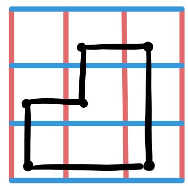




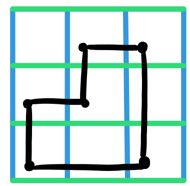
turn 45° clockwise ⇝ Legendrian front

SURFACES IN $\mathbb{CP}^2$

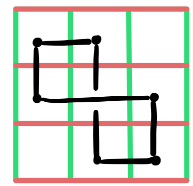




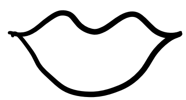
blue over red

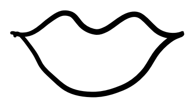


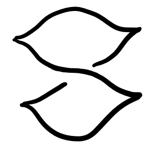
green over blue



red over green







turn 45° clockwise ⇝ Legendrian front

if all three Legendrian links are tb-1 unlinks, can ~~cap~~ fill off w/ Lagrangian disks ⇝ embedded Lagrangian-like surface in $\mathbb{CP}^2$

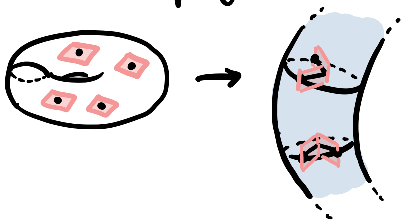
SURFACES IN $\mathbb{CP}^2$

Theorem [B.] * Lagrangian-like surfaces in $\mathbb{CP}^2$ are smoothly isotopic to Lagrangians in $\mathbb{CP}^2$

SURFACES IN $\mathbb{CP}^2$

Theorem [B.] * Lagrangian-like surfaces in $\mathbb{CP}^2$ are smoothly isotopic to Lagrangians in $\mathbb{CP}^2$

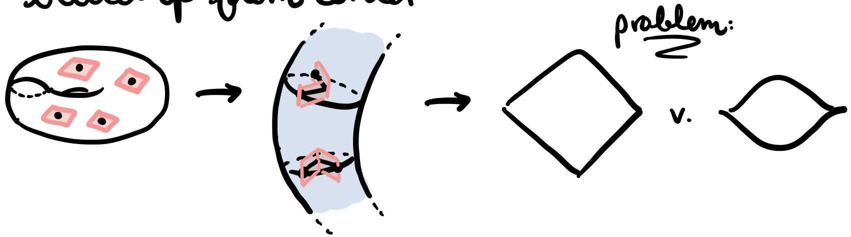
idea: build up from center



SURFACES IN $\mathbb{CP}^2$

Theorem [B.] * Lagrangian-like surfaces in $\mathbb{CP}^2$ are smoothly isotopic to Lagrangians in $\mathbb{CP}^2$

idea: build up from center

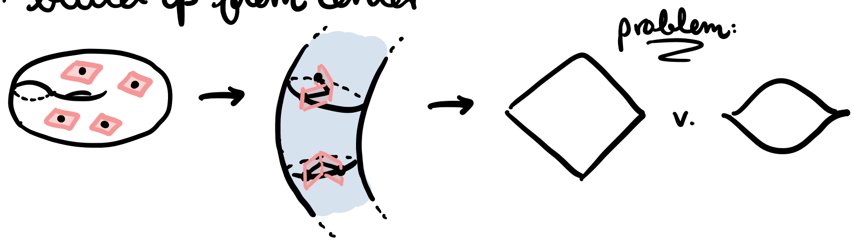


need to smooth corners

SURFACES IN $\mathbb{CP}^2$

Theorem [B.]* Lagrangian-like surfaces in $\mathbb{CP}^2$ are smoothly isotopic to Lagrangians in $\mathbb{CP}^2$

idea: build up from center



need to smooth corners

question do TKGD uniquely determine Lagrangians? (up to what?)

question is every Lagrangian in $\mathbb{CP}^2$ (Hamiltonian)(Lagrangian) isotopic to one given by a triple grid diagram?

EVIDENCE + BEHAVIOR

proposition [B.] if all of the faces of a TKGD are squares, then the surface is $\mathbb{R}P^2$, and in particular the diagram must have one face in every grid

proposition [B.] if all of the faces of a TKGD are squares (4 vertices) or hexagons (6 vertices) then the surface has Euler characteristic 0, 1, or 2

(+ many examples; only orientable (embedded) Lagrangian-like examples are tori)

CLASSIFICATION

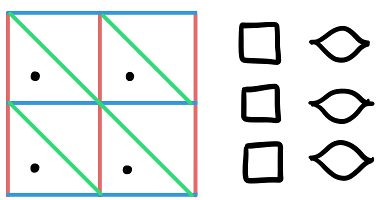
$\boxed{\text{Theorem [B.]}}$ for full (no empty rows, columns, diagonals) TKGDs:

- 2×2: unique diagram which is a Lagrangian-like $\mathbb{RP}^2$
- 3×3: unique diagram which is a Lagrangian-like T^2
- 4×4: three diagrams which are all $\mathbb{RP}^2$; two are Lagrangian-like
- 5×5: six diagrams; four are $\#^2 T^2$; two are $\#^4 \mathbb{RP}^2$; none are Lagrangian-like

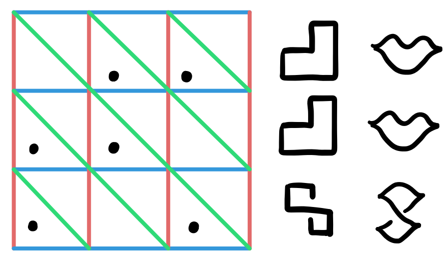
CLASSIFICATION

Theorem [B.] for full (no empty rows, columns, diagonals) TKGDs:

- 2×2: unique diagram which is a Lagrangien-like $\mathbb{RP}^2$
- 3×3: unique diagram which is a Lagrangien-like T^2
- 4×4: three diagrams which are all $\mathbb{RP}^2$; two are Lagrangian-like
- 5×5: six diagrams; four are $\#^2 T^2$; two are $\#^4 \mathbb{RP}^2$; none are Lagrangian-like

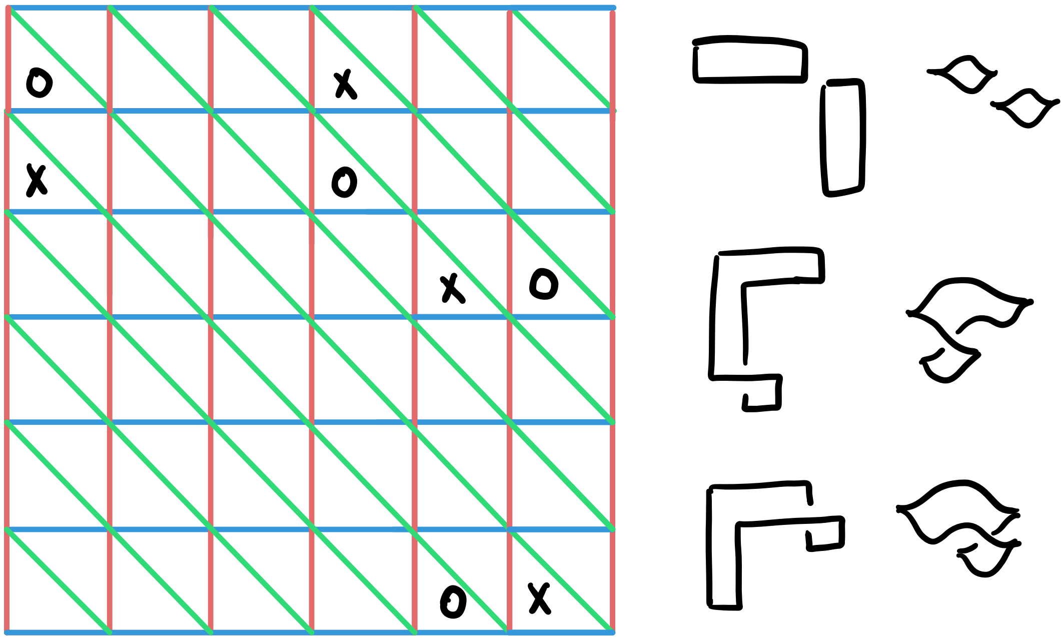


$\mathbb{RP}^2$



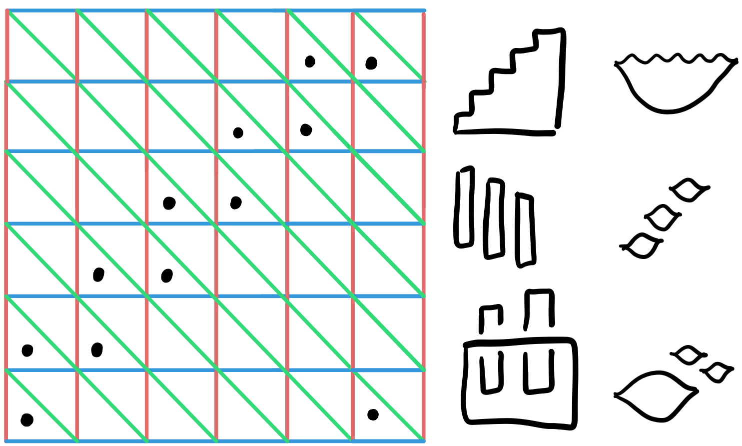
T^2

CLASSIFICATION

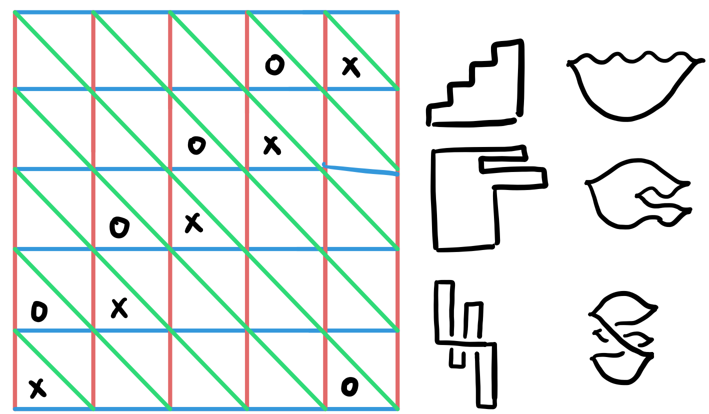


another Lagrangian-like torus

CLASSIFICATION



n even: Lagrangian-like $\mathbb{RP}^2$



n odd: $\#_{\frac{n-1}{2}} T^2$

(Lagrangian-like for n=3)

MINIMAL TRIPLE GRID

n×n

- of a smooth link L:

  smallest n st ∃ an n×n triple grid diagram with L as one of its links

- of a Legendrian link L:

  smallest n st ∃ an n×n triple grid diagram with L as one of its links

- of a triple of Legendrian links L_1, L_2, L_3:

  smallest n st ∃ an n×n triple grid diagram with L_1, L_2, L_3 as its links

...

MINIMAL TRIPLE GRID

nxn

- of a smooth link L:

  smallest n st ∃ an nxn triple grid diagram with L as one of its links

- of a Legendrian link L:

  smallest n st ∃ an nxn triple grid diagram with L as one of its links

- of a triple of Legendrian links L_1, L_2, L_3:

  smallest n st ∃ an nxn triple grid diagram with L_1, L_2, L_3 as its links

KNOWN RESULTS

the minimal triple grid number of...

the smooth unknot is 2

the smooth two component unlink is 4

the smooth trefoil is 5

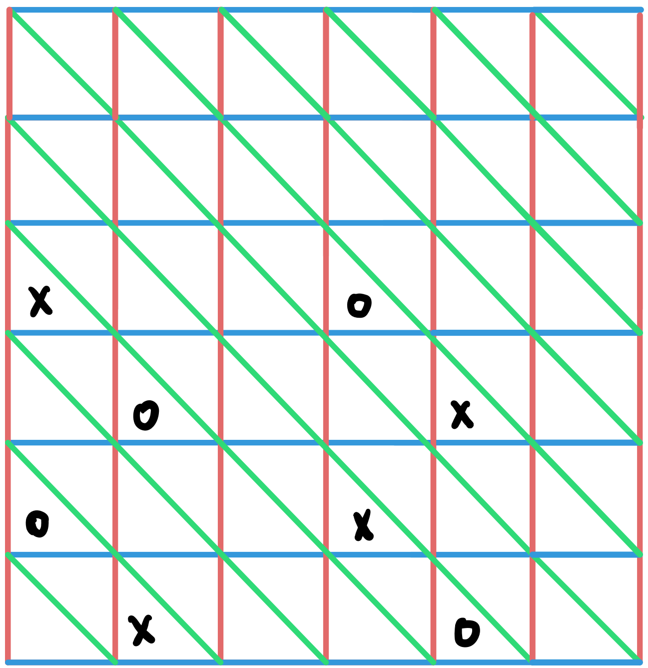
the smooth Hopf link is 6

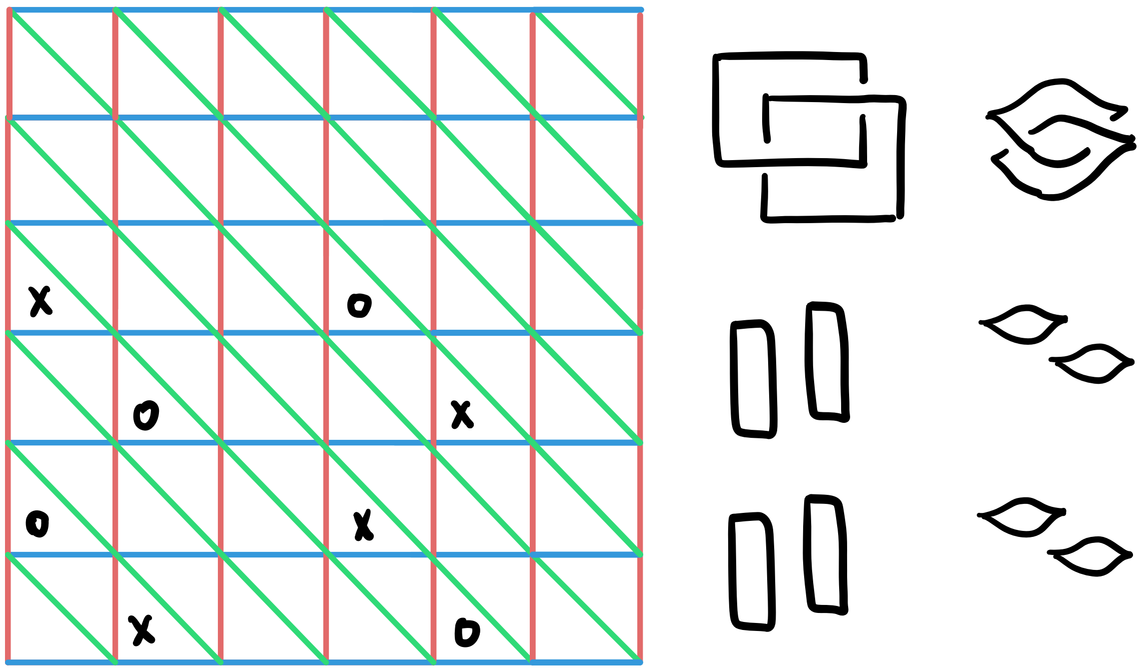
the smooth three component unlink is 6

the Legendrian max tb unknot is 2

a triple of Legendrian max tb unknots is 2

the Hopf link with max tb unknot components is 6





Lagrangian-like S^2 w/ double pt

THANKS!