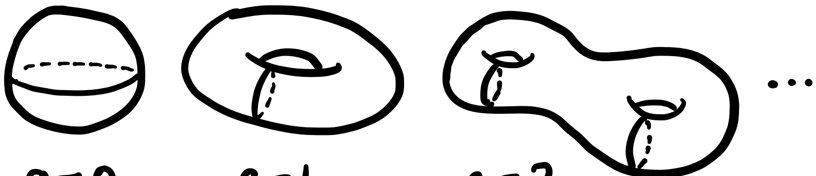
GROUP TRISECTIONS + SMOOTHLY KNOTTED SURFACES

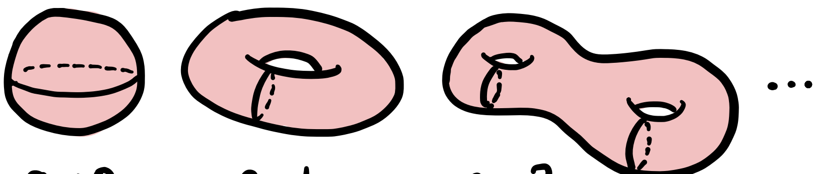
Sarah Blackwell
University of Georgia
j.w. Kirby, Klug, Longo, Ruppik

Σ_g = genus g surface



$g=0$ $g=1$ $g=2$

H_g = genus g handlebody



$g=0$ $g=1$ $g=2$

* everything is smooth! *

BIG PICTURE

topology		algebra
trisected 4-mfld X	apply van Kampen thm to pieces $\longrightarrow$	group trisection of $\pi_1(X)$
	$\longleftarrow$ abrams, Gay, Kirby	

* everything is smooth! *

BIG PICTURE

topology

algebra

trisected 4-mfld X

apply van Kampen thm to pieces

abrams, Gay, Kirby

group trisection of $\pi_1(X)$

inherited from

bridge trisected surface S

everything is smooth!

BIG PICTURE

topology | **algebra**

trisected 4-mfld X $\xrightarrow{\text{apply van Kampen thm to pieces}}$ group trisection of $\pi_1(X)$

$\longleftarrow$ Abrams, Gay, Kirby

↑ inherited from

bridge trisected surface S $\xrightarrow{\text{apply van Kampen thm to pieces}}$ group trisection of $\pi_1(X \setminus S)$

$\longleftarrow$ B.K.K.L.R.

* everything is smooth! *

BIG PICTURE

topology

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trisected 4-mfld X

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bridge trisected surface S

apply van Kampen thm to pieces

B.K.K.L.R. *

group trisection of $\pi_1(X \setminus S)$

* precise statement in forthcoming paper

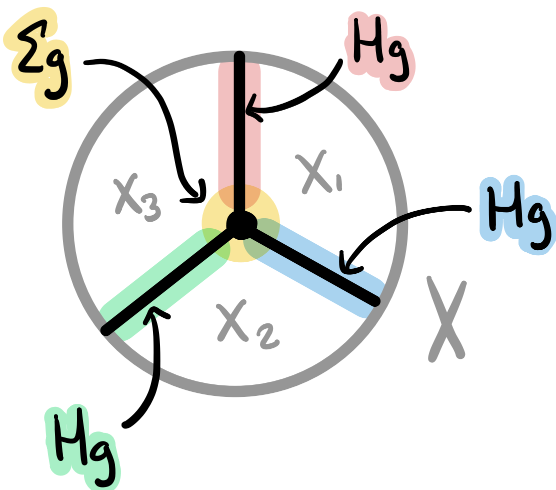
π_1 of knotted surface complement in S^4 does not determine the knotted surface, but the trisection of π_1 does determine it

TRISECTIONS

$\underline{(g,k)\text{- trisection}}$ of smooth, closed, connected, oriented 4-mfld X = decomposition $X = X_1 \cup X_2 \cup X_3$ st

[Gay + Kirby]

1) $X_i \cong \natural^k (S^1 \times B^3)$ = 4D 1-handlebody
2) $X_i \cap X_j \cong H_g$ = genus g handlebody
3) $X_1 \cap X_2 \cap X_3 \cong \Sigma_g$ = genus g surface



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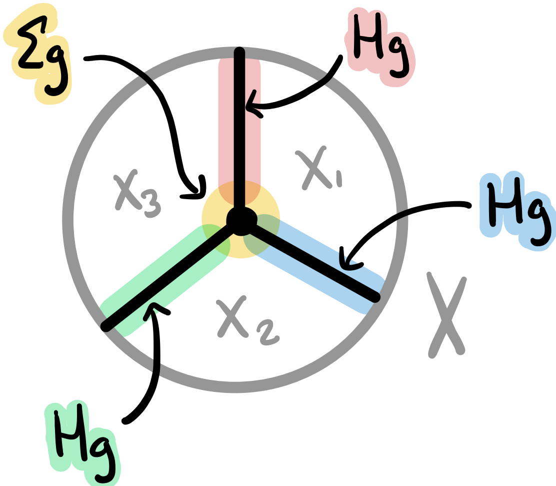
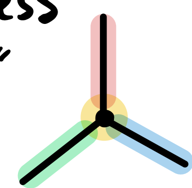


- existence + uniqueness
- determined by "spine"



GROUP TRISECTIONS

(g,k)-group trisection of a group G [Abrams, Gay, Kirby] = commutative cube of groups (as shown below) st every homomorphism is surjective and each face is a pushout

$$
\begin{tikzcd}
& \pi_1(H_g) \arrow[rr, two heads] \arrow[dd, dashed, two heads] & & \pi_1(X_2) \arrow[dd, two heads] \\
\pi_1(\Sigma_g) \arrow[ur, two heads] \arrow[rr, two heads] \arrow[dd, two heads] & & \pi_1(H_g) \arrow[ur, two heads] \arrow[dd, two heads] & \\
& \pi_1(X_3) \arrow[rr, dashed, two heads] & & \pi_1(X) \\
\pi_1(H_g) \arrow[ur, dashed, two heads] \arrow[rr, two heads] & & \pi_1(X_1) \arrow[ur, two heads] &
\end{tikzcd}
$$

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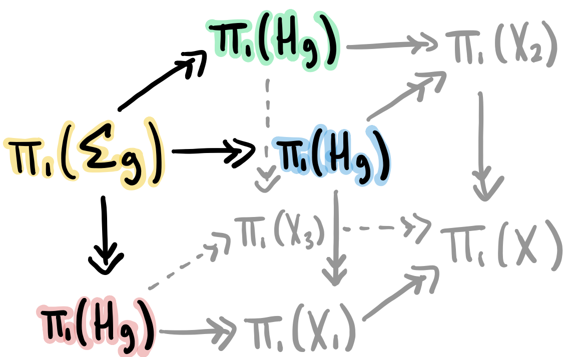
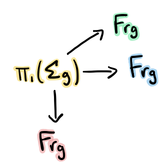


- existence + uniqueness
- determined by "tripod"

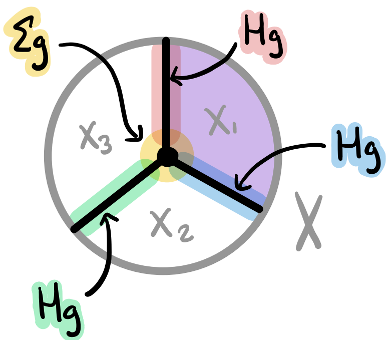


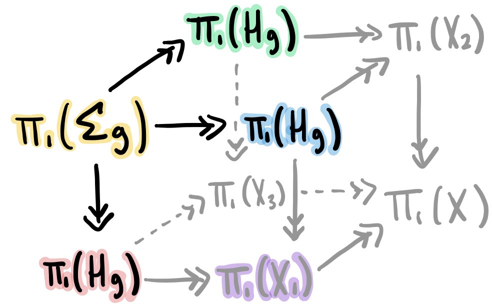
topology

algebra

trisected 4-mfld X $\xrightarrow{\text{apply van Kampen thm to pieces}}$ group trisection of $\pi_1(X)$

$\xleftarrow{\text{abrams, Gay, Kirby}}$





topology

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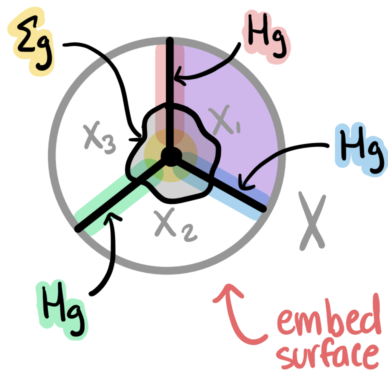
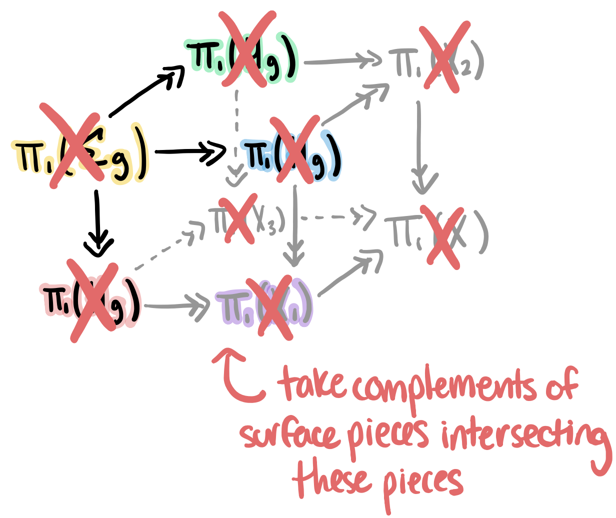
$\xleftarrow[\text{Abrams, Gay, Kirby}]{}$





topology

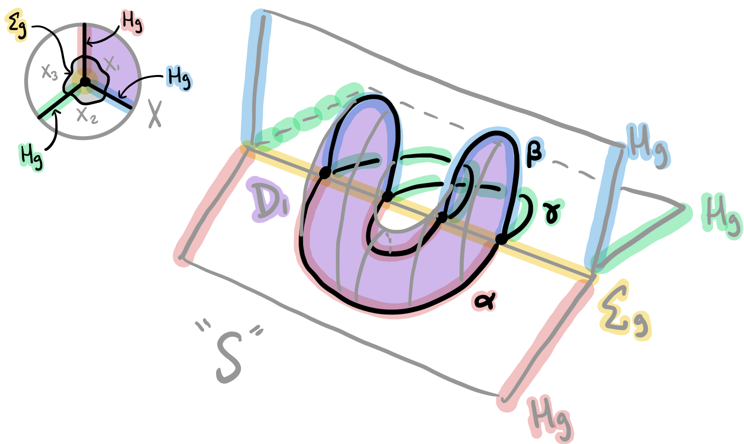
algebra

bridge trisected surface S

apply van Kampen thm to pieces $\longrightarrow$

$\longleftarrow$ B.K.K.L.R.

group trisection of $\pi_1(X \setminus S)$



[Meier + Zupan]

$$\begin{tikzcd}
 & \pi_1(H_g \setminus \gamma) \arrow[rr, two heads] \arrow[dd, dashed, two heads] & & \pi_1(X_2 \setminus D_2) \arrow[dd] \\
\pi_1(\Sigma_g \setminus pts) \arrow[ur] \arrow[rr, two heads] \arrow[dd, two heads] & & \pi_1(H_g \setminus \beta) \arrow[ur, two heads] \arrow[dd, two heads] & \\
 & \pi_1(X_3 \setminus D_3) \arrow[rr, dashed] & & \pi_1(X \setminus S) \\
\pi_1(H_g \setminus \alpha) \arrow[ur, dashed, two heads] \arrow[rr, two heads] & & \pi_1(X_1 \setminus D_1) \arrow[ur, two heads] &
\end{tikzcd}$$

how to get a surface from this?

$$
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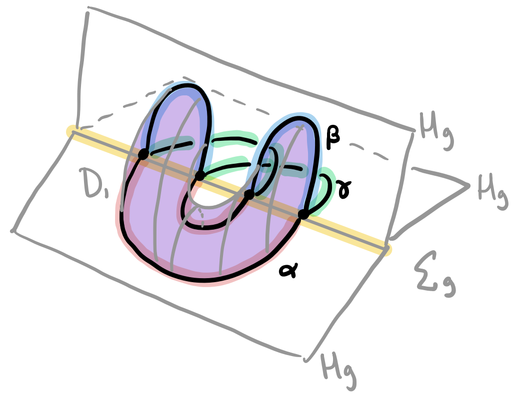
\end{tikzcd}
\quad \rightsquigarrow
$$



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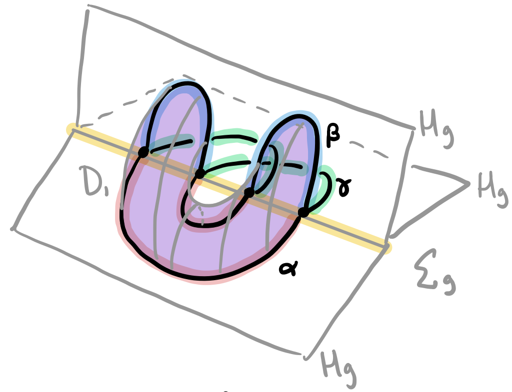

only need to get 3 tangles

$$\begin{tikzcd}
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& & \\
\pi_1(H_g \setminus \alpha) & &
\end{tikzcd}$$

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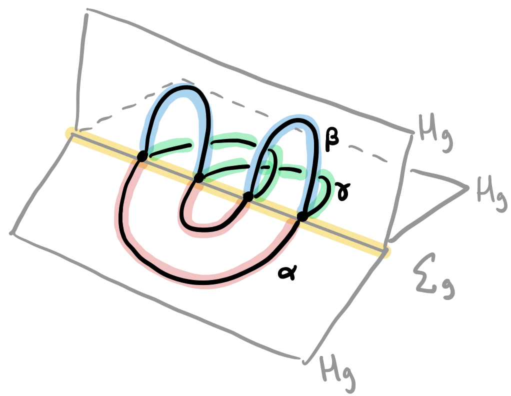

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$$\begin{tikzcd}
 & \pi_1(H_g \setminus \gamma) \arrow[rr, two heads] \arrow[dd, dashed, two heads] & & \pi_1(X_2 \setminus D_2) \arrow[dd, two heads] \\
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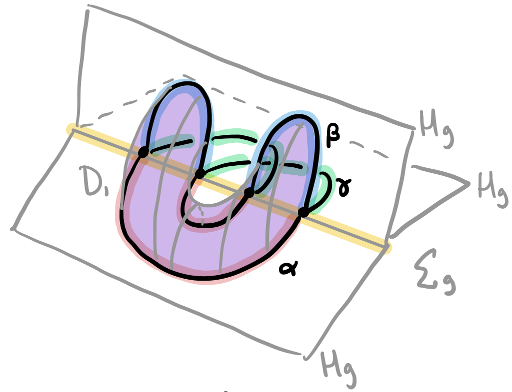

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 & & \\
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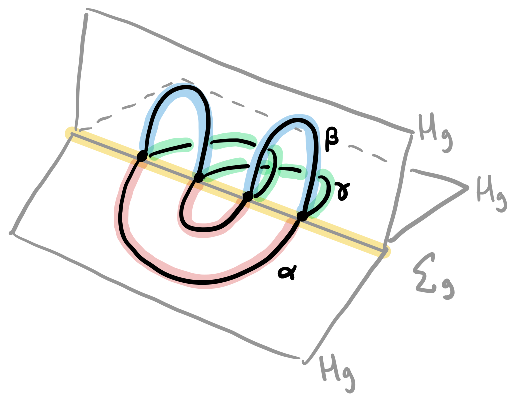
$\rightsquigarrow$



(or, get <u>a</u> tangle, 3 times)

how to get a surface from this?

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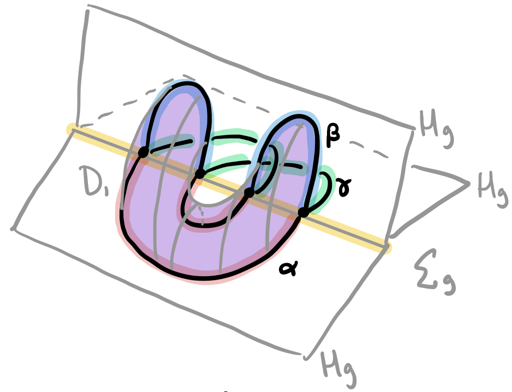
$\rightsquigarrow$



only need to get 3 tangles

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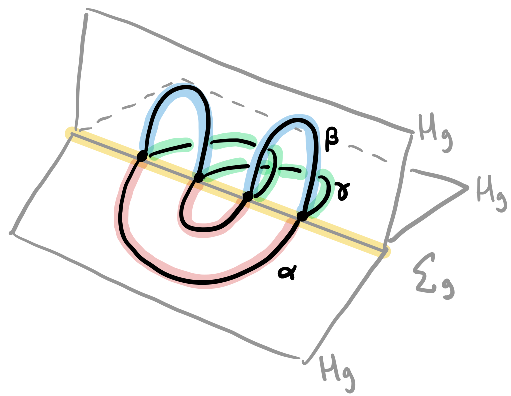
\pi_1(H_g \setminus \alpha) &
\end{tikzcd}$$

all free groups!

$\rightsquigarrow$



(or, get $\underline{\underline{a}}$ tangle, 3 times)

EXISTENCE + UNIQUENESS

EXISTENCE + UNIQ~~U~~ENESS

now

see paper

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#1 algorithm: surjection btw free gps $\rightsquigarrow$ tangle in a handlebody

EXISTENCE + UNI~~QU~~ENESS

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via cancellation in surface relation

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#2 choice of cancellation doesn't matter

get rid of extra closed components via band sums

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see paper

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#3 bands always exist

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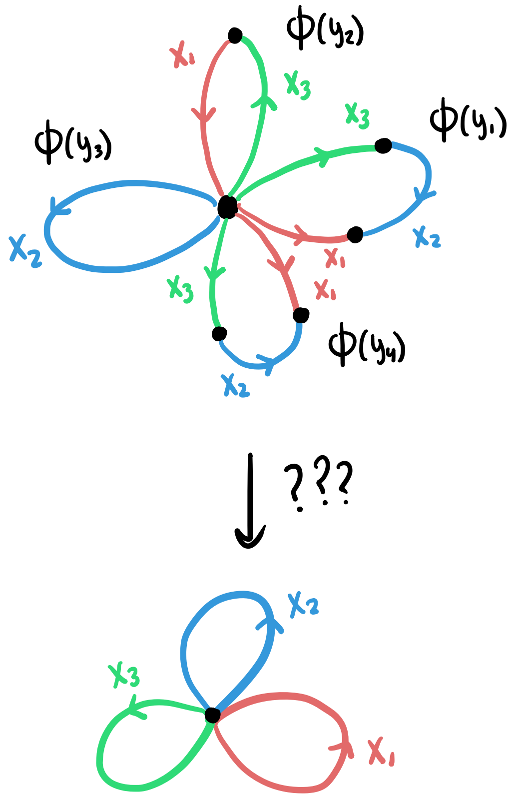
Stallings folding guides these parts

STALLINGS FOLDING

determines whether a map btw free gps is surj.



$F_4 \xrightarrow{\phi} F_3$ (e.g.)

$\phi(y_1) = x_1 \bar{x}_2 \bar{x}_3$

$\phi(y_2) = x_3 x_1$

$\phi(y_3) = x_2$

$\phi(y_4) = x_3 x_2 \bar{x}_1$

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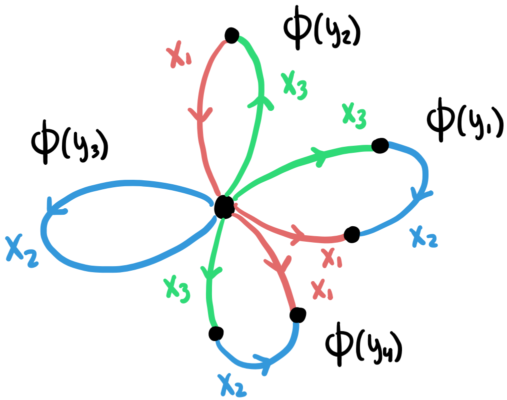

$$F_4 \xrightarrow{\phi} F_3 \quad \text{(e.g.)}$$

$$\phi(y_1) = x_1 \bar{x}_2 \bar{x}_3$$
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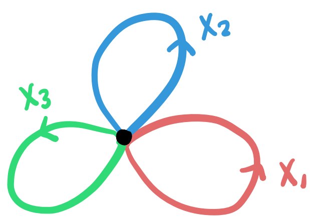
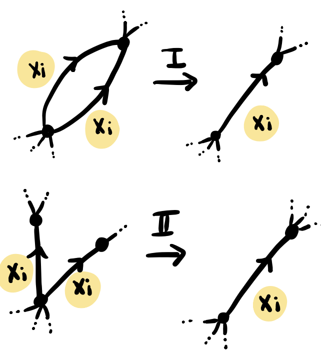
$$\phi(y_4) = x_3 x_2 \bar{x}_1$$

$\downarrow$???

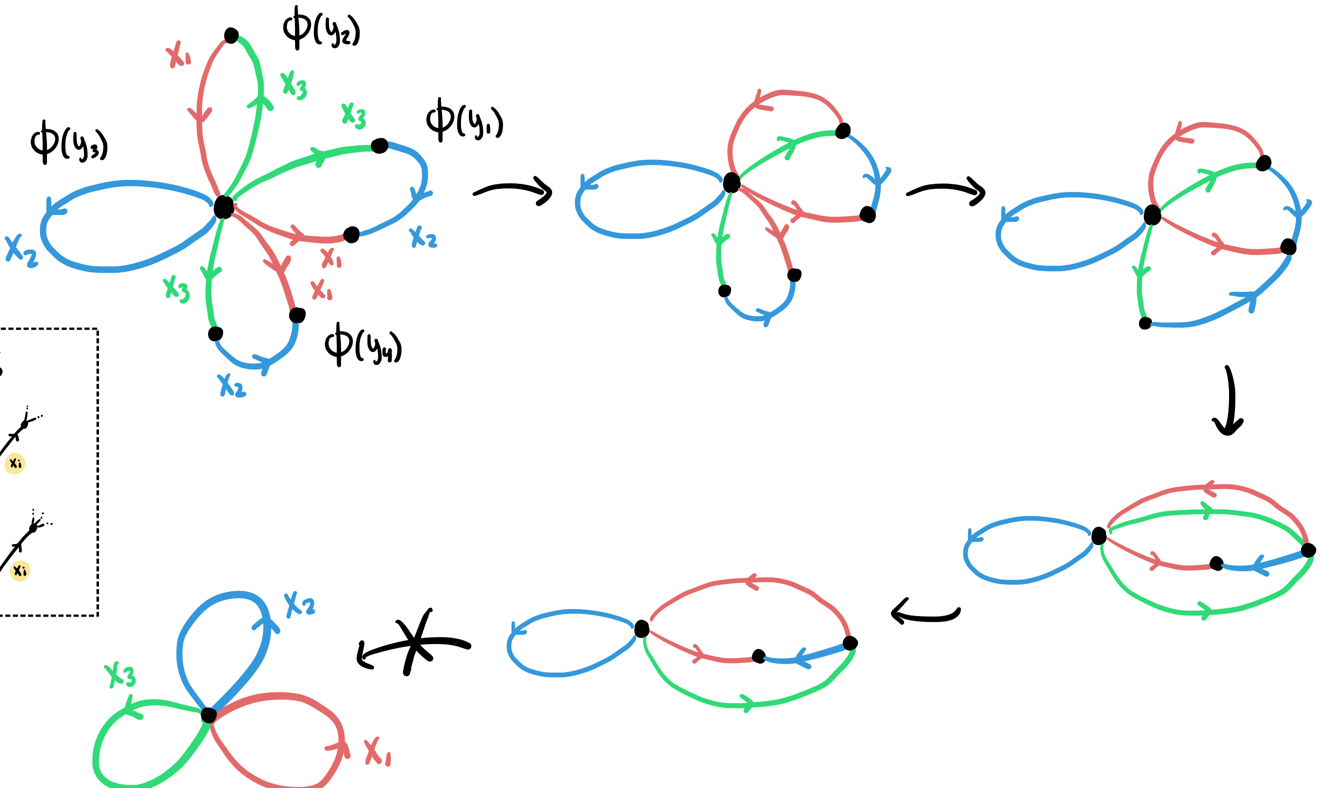


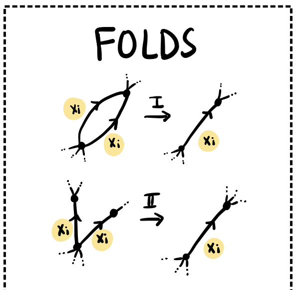
FOLDS



STALLINGS FOLDING

determines whether a map btw free gps is surj.

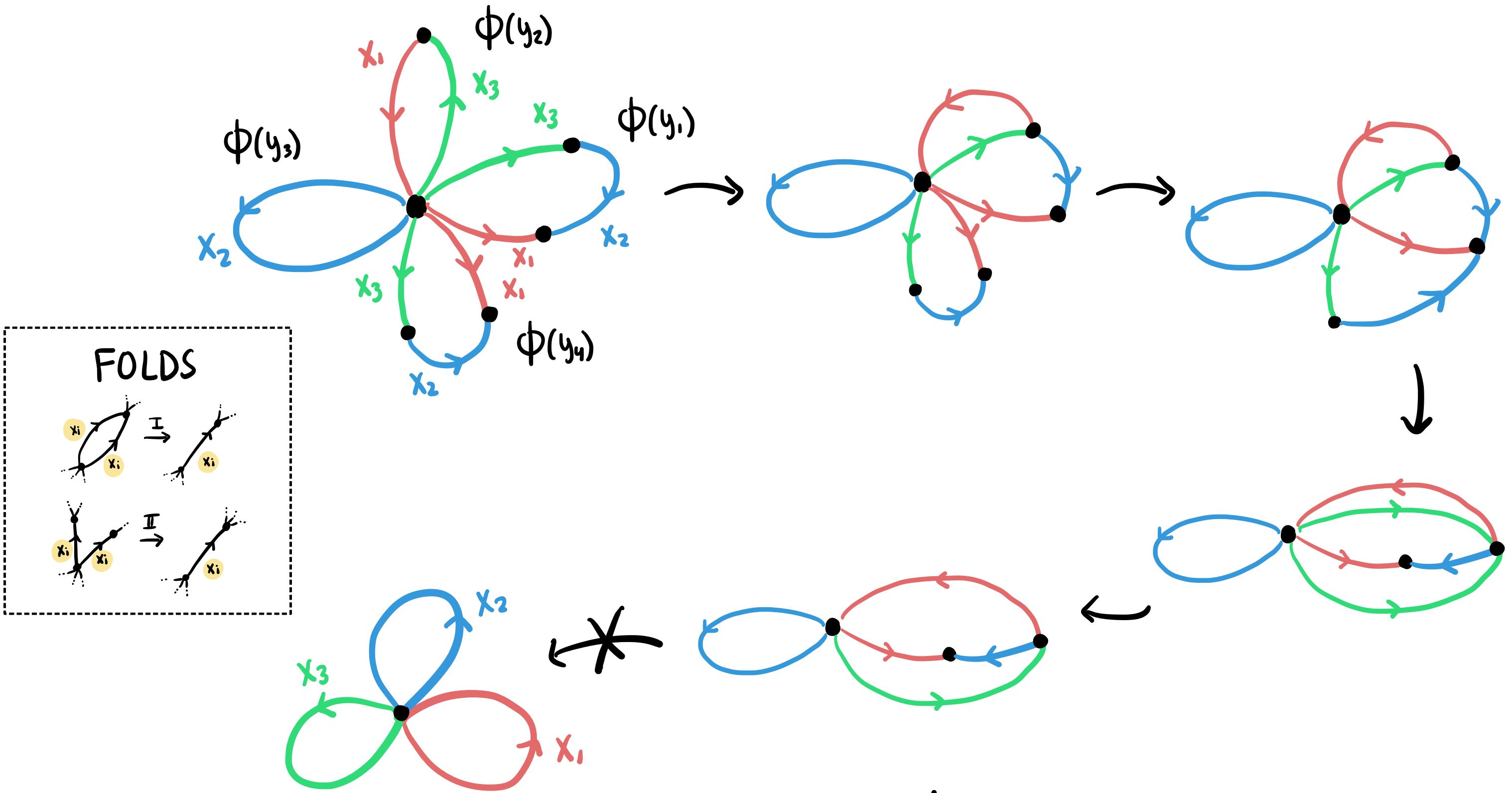




∴ ϕ is not surjective

STALLINGS FOLDING

determines whether a map btw free gps is surj. (iff!)



$\therefore$ ϕ is not surjective

EXISTENCE + ~~UNIQUENESS~~

now

see paper

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That's all Folks!