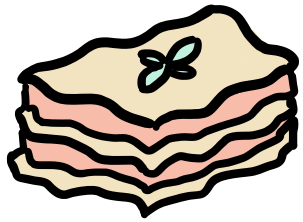

Cornered Skein Lasagna Theory

Sarah Blackwell
University of Virginia

j.w. Slava Krushkal + Yangxiao Luo

antipasto:

OVERVIEW

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[2019, Morrison-Walker-Wedrich] "skein lasagna module"

"what if we could do Khovanov homology in 4D?"

OVERVIEW

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POWERFUL	HARD TO CALCULATE

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OVERVIEW

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- $S^2 \times S^2$ [2024, Sullivan-Zhang] (generalization in [RW])

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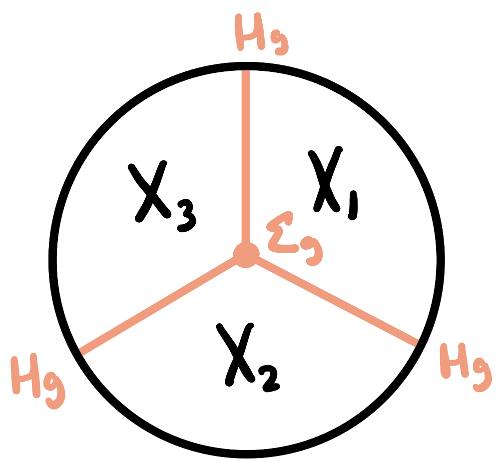

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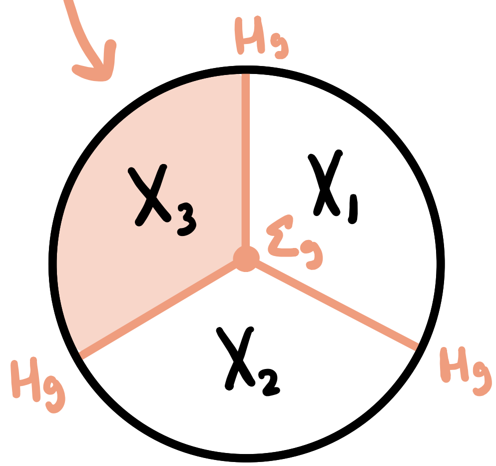

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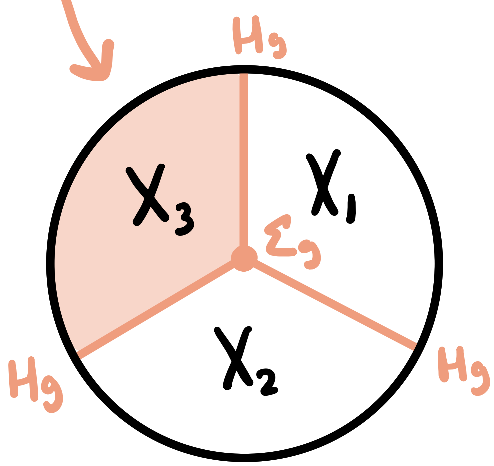

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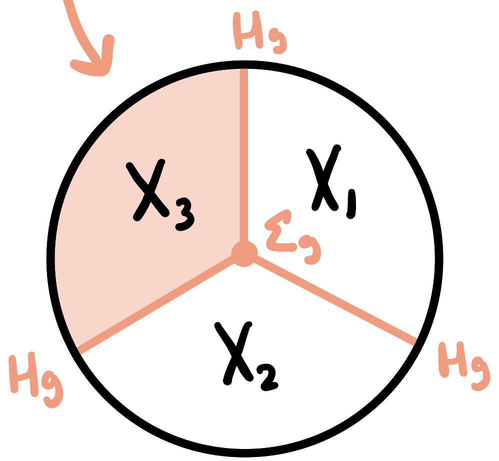

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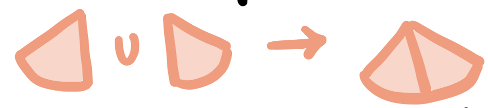
our work:

(stated/proved more generally in preprint)

- develop categorical framework
- define cornered skein lasagna invariant
- gluing theorem 1: two pieces along common boundary



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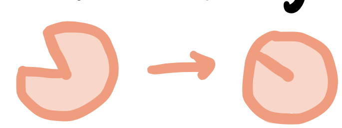

primo:

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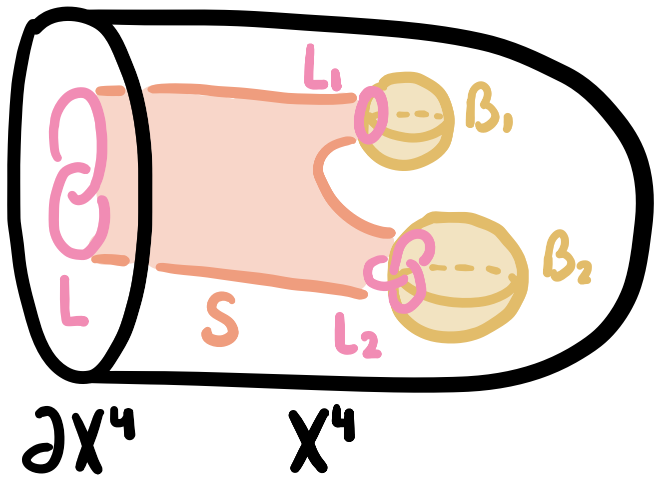

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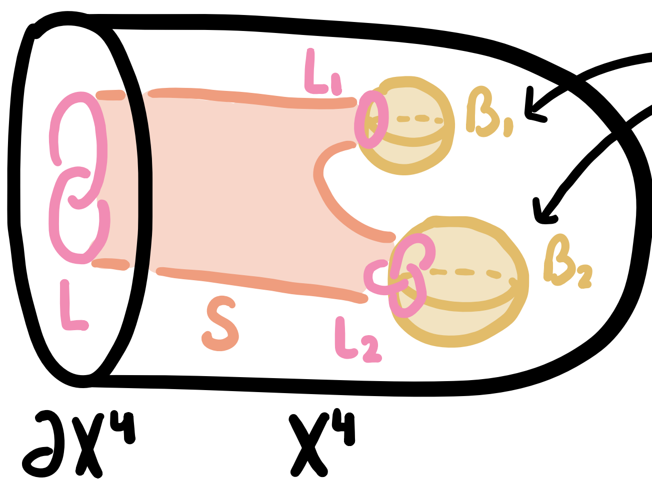

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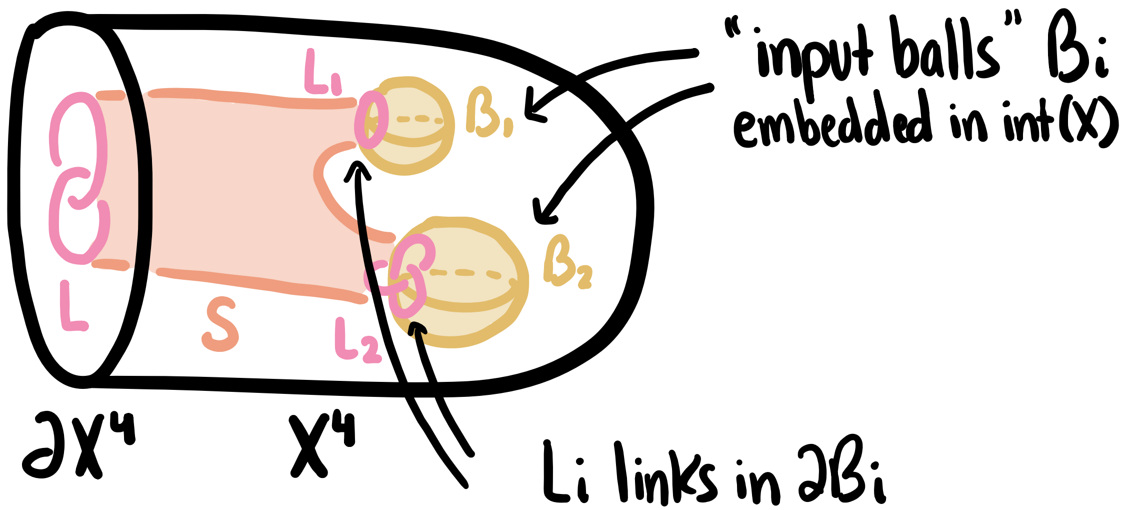

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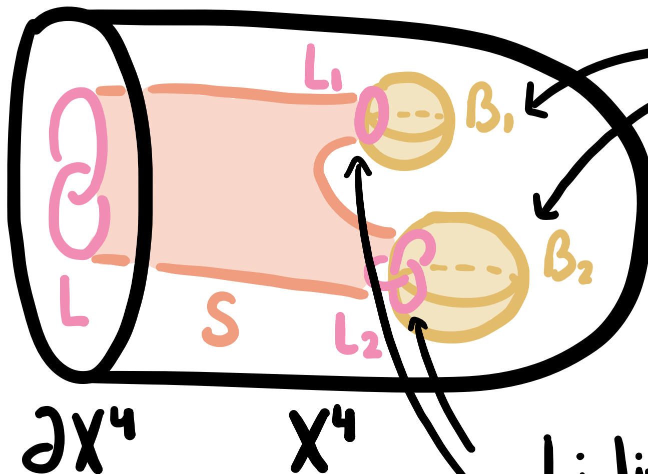

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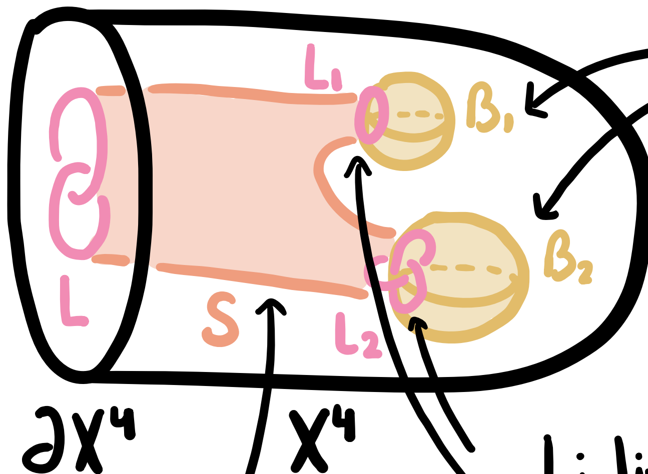

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oriented surface S
properly embedded in $X \setminus \cup B_i$
st $S \cap \partial X = L$ and $S \cap \partial B_i = L_i$

LASAGNAS

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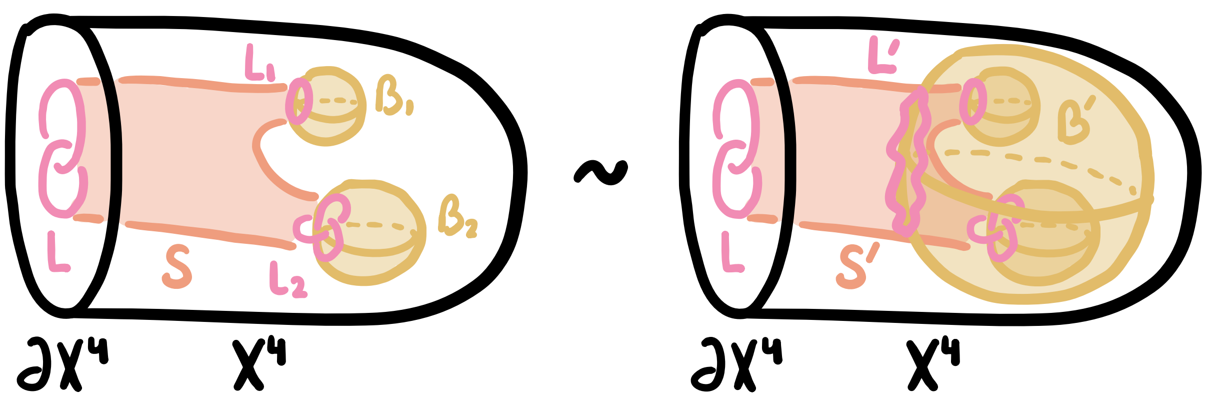

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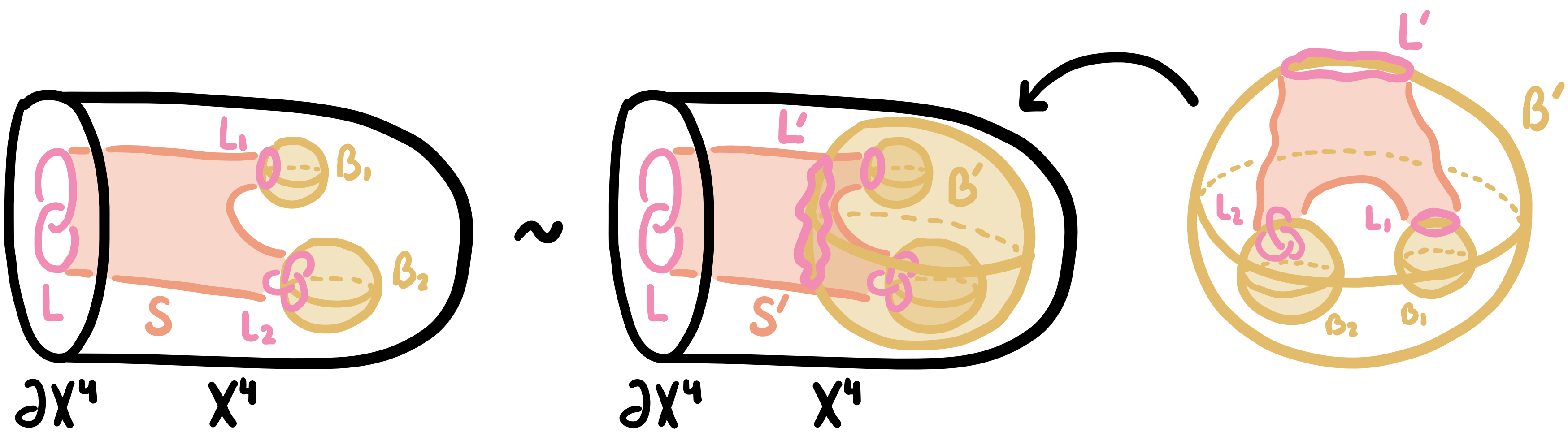

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secondo:

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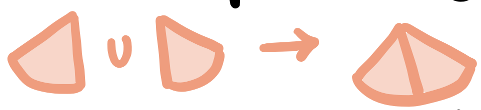

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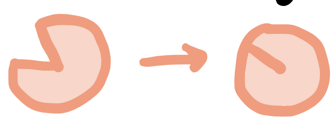

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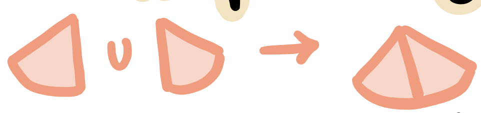

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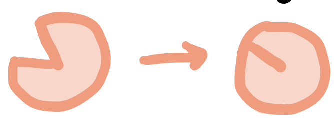

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Theorem [BKL]: the map $S(X_1, -) \underset{S(Y_2, P)}{\otimes} S(X_2, -) \longrightarrow S(X_1 \cup X_2, -)$ is an isomorphism

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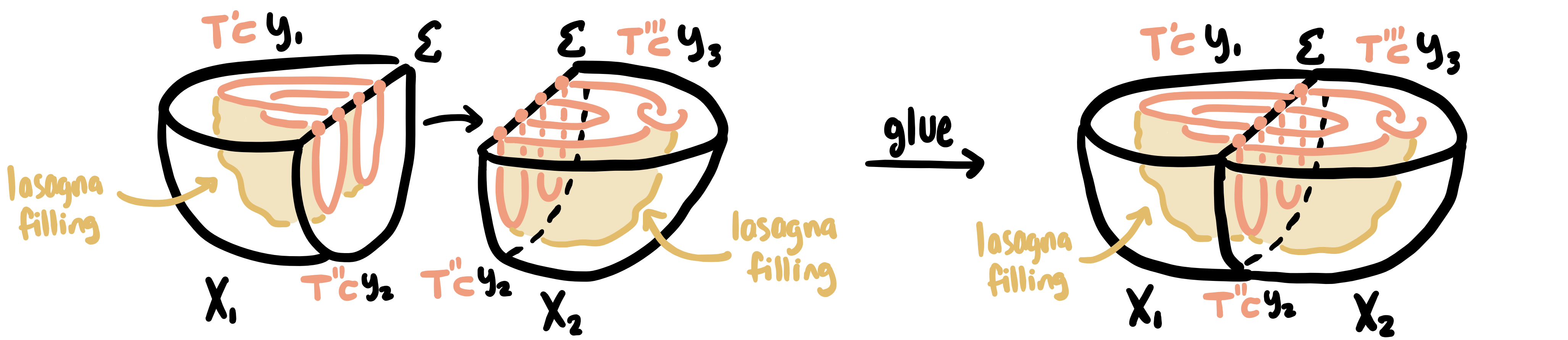
???

$T' \cup T''$?? $T'' \cup T'''$ $T' \cup T'''$



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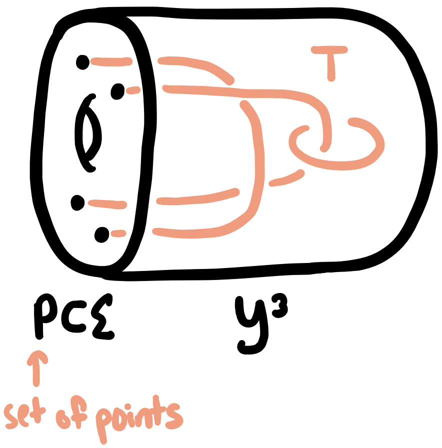

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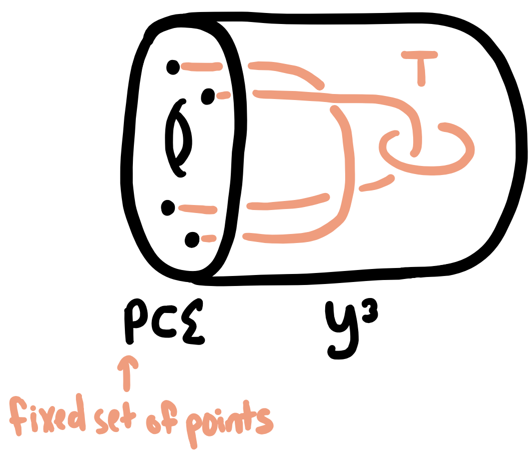

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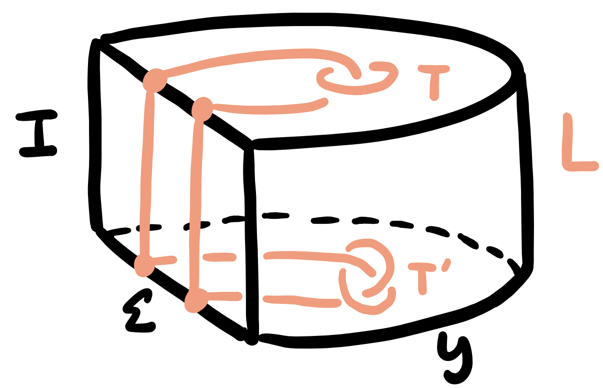
Obj = tangles T properly embedded in Y st $\partial T = P$

$\mathrm{Mor}(T,T') = S(Y \times I, L)$ skein lasagna module



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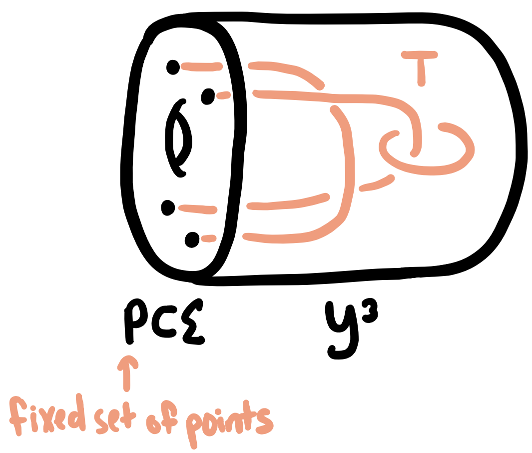

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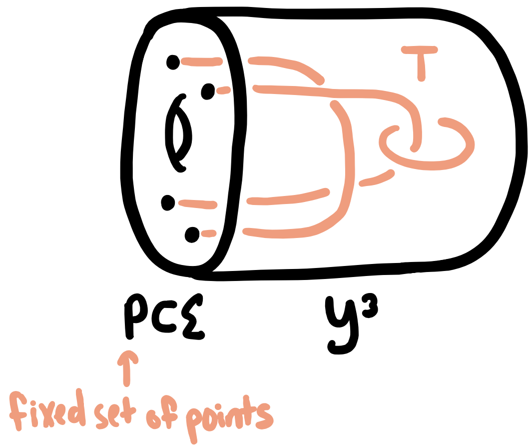

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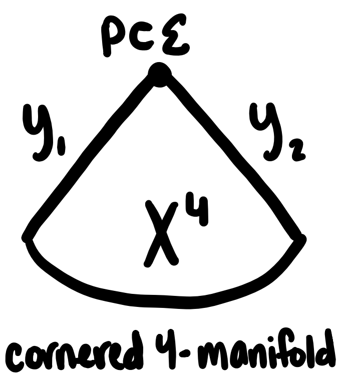
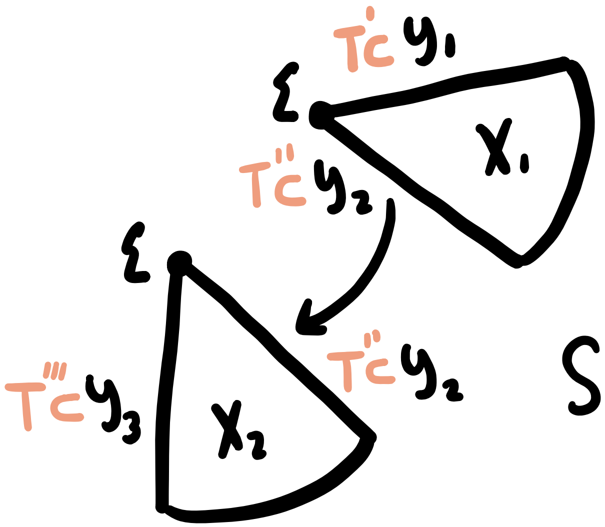


$\rightsquigarrow$ "cornered skein lasagna bimodule" which is a bifunctor

$S(X,-) : S(Y_1,P) \times S(Y_2,P) \longrightarrow \mathcal{V}$

category of $\mathbb{Z}^2$-graded $\mathbb{Z}$-modules

OUR WORK



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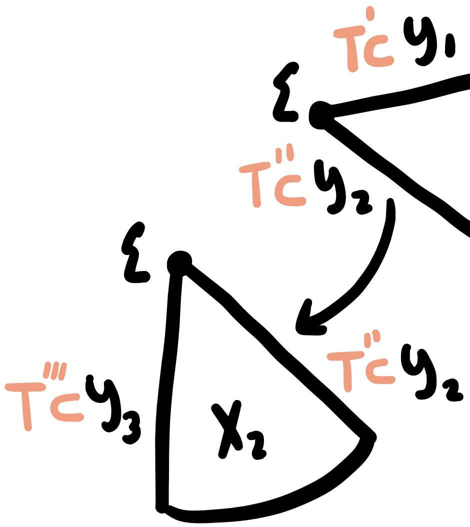

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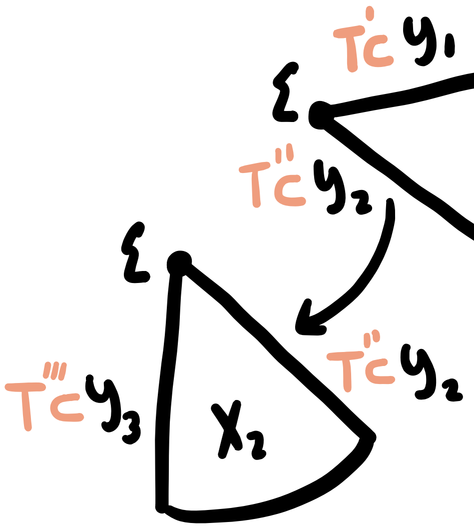

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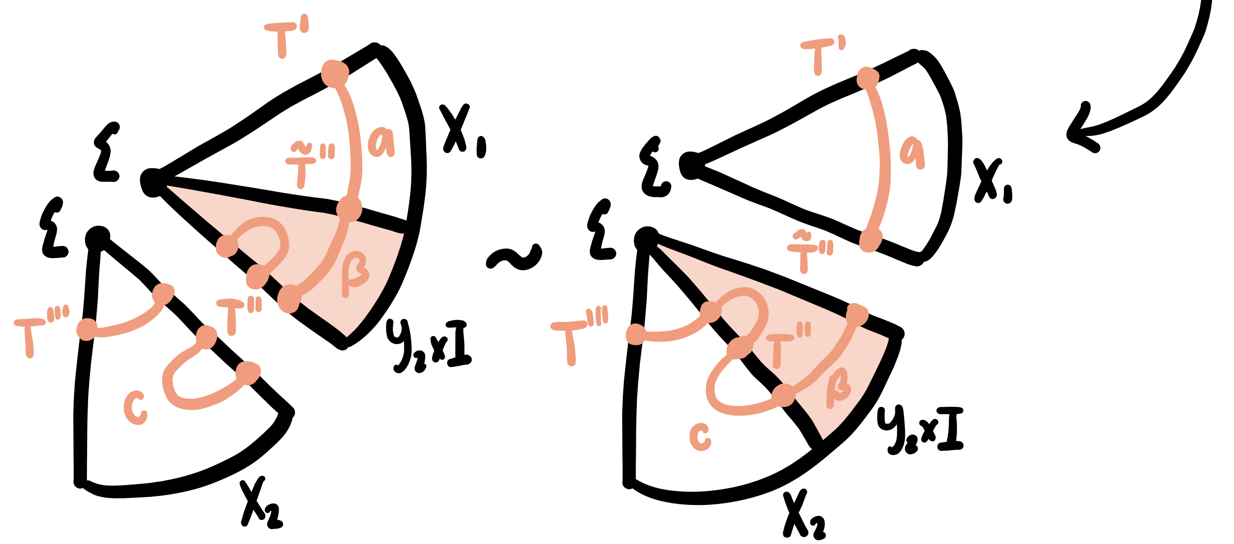

dolce :

CONCRETE CALCULATIONS

(TBD)

THANKS!