CLASSICAL INVARIANTS OF SPIRAL KNOTS

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KNOTS

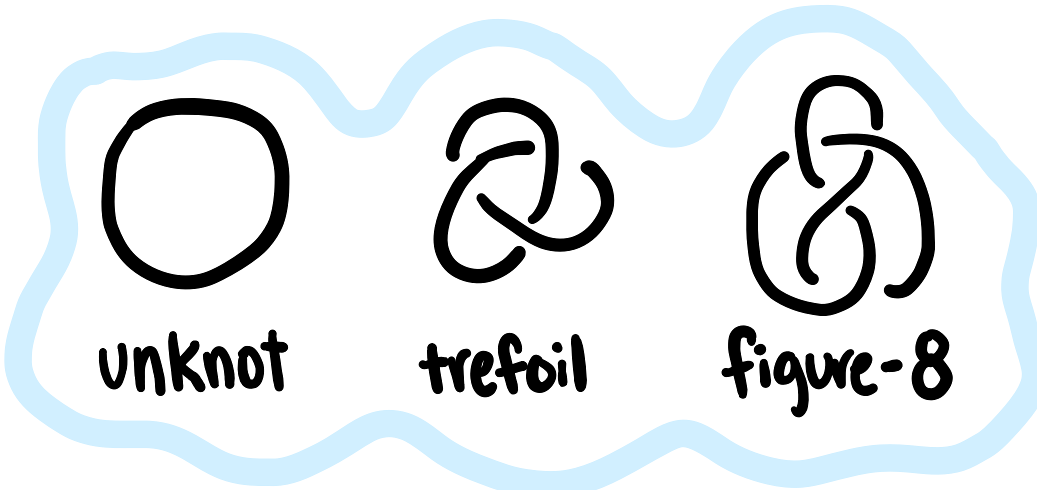
$=$

smooth embedding $S^1 \hookrightarrow \mathbb{R}^3$ or S^3 ...

KNOTS

$\shortparallel$

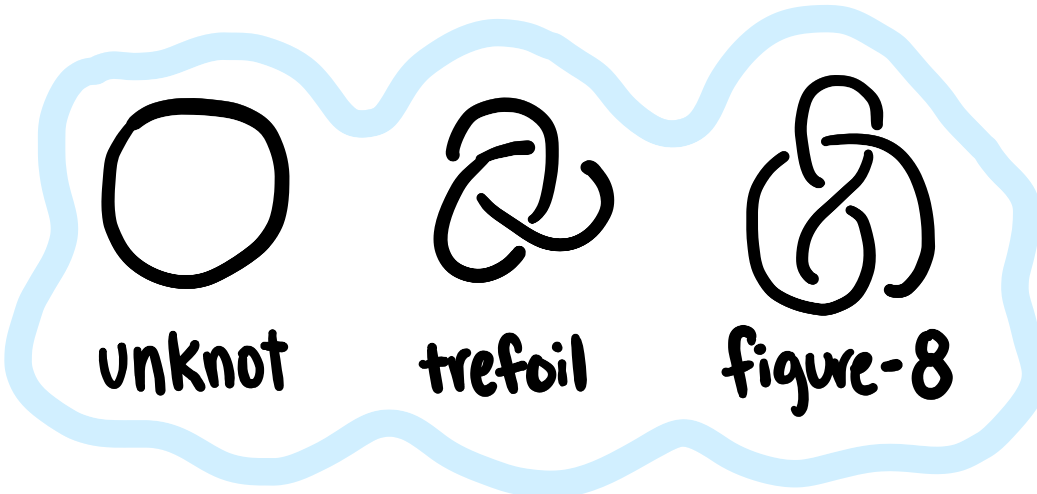
smooth embedding $S^1 \hookrightarrow \mathbb{R}^3$ or S^3 ...



KNOTS

=

smooth embedding $S^1 \hookrightarrow \mathbb{R}^3$ or S^3 ...



... considered up to (ambient) isotopy

KNOTS

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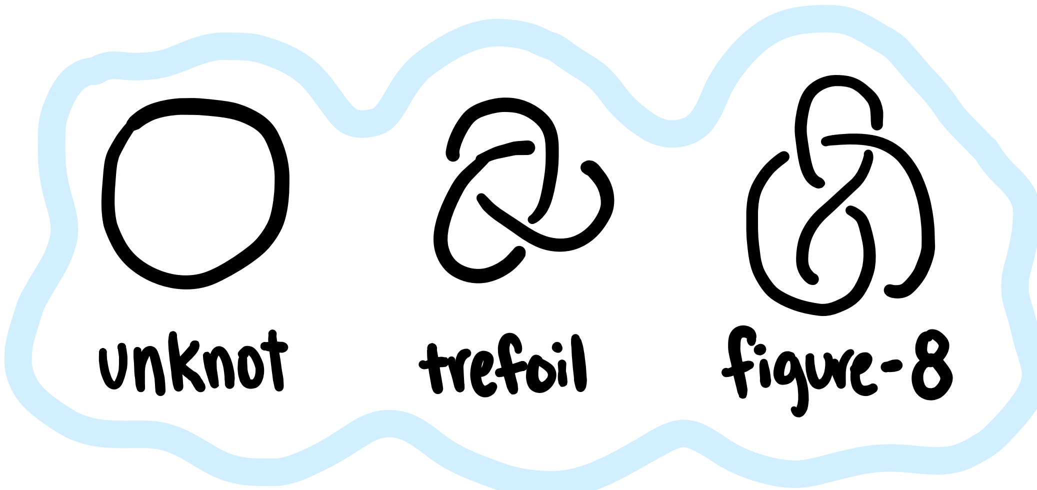
smooth embedding $S^1 \hookrightarrow \mathbb{R}^3$ or S^3 ...



... considered up to (ambient) isotopy

$K_1 = K_2$ if you can wiggle K_1 in $\mathbb{R}^3$ or S^3 to be K_2 w/o "cutting"

KNOTS

$\parallel$

smooth embedding $S^1 \hookrightarrow \mathbb{R}^3$ or S^3 ...

unknot   trefoil   figure-8

a knot diagram $= \pi(S^1) \subset \mathbb{R}^2$, where $\pi : \mathbb{R}^3 \to \mathbb{R}^2$

KNOTS

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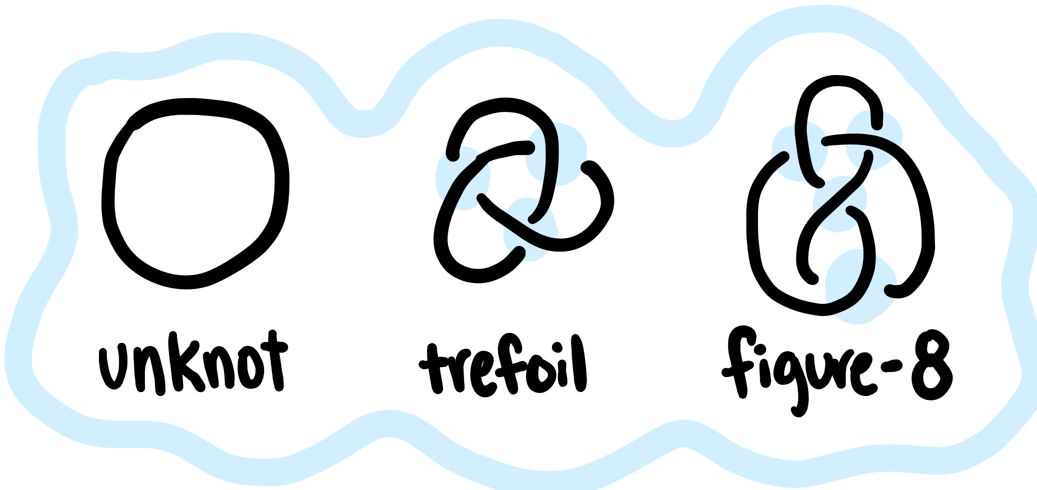
smooth embedding $S^1 \hookrightarrow \mathbb{R}^3$ or S^3 ...

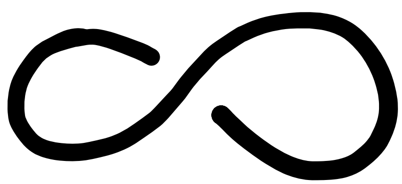


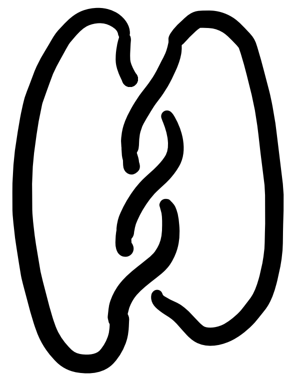
a Knot diagram = $\pi(S^1) \subset \mathbb{R}^2$, where $\pi : \mathbb{R}^3 \to \mathbb{R}^2$

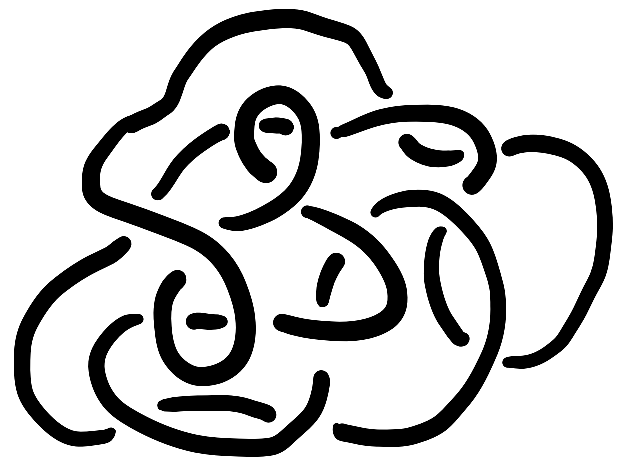
usually, $\pi(S^1)$ will have double points which we call crossings

KNOTS

diagrams are not unique :







KNOTS

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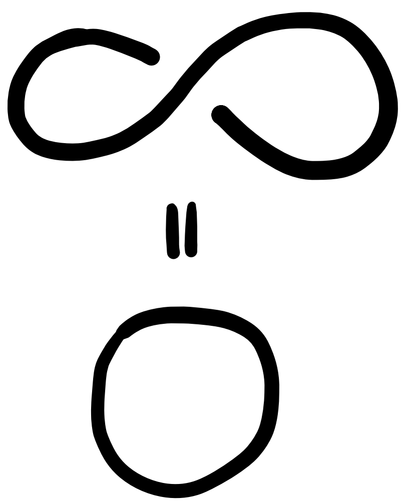
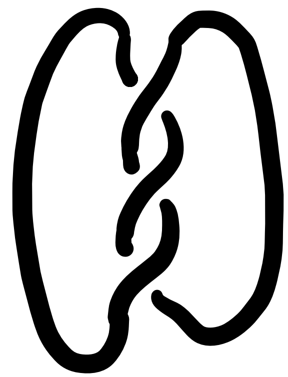
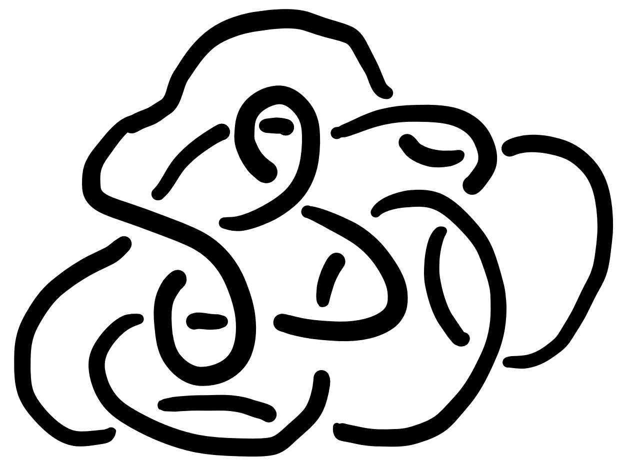


unknot





KNOTS

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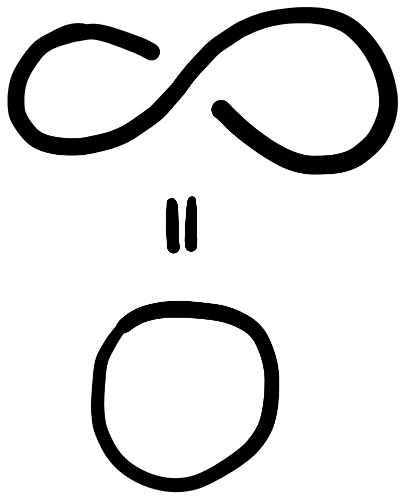
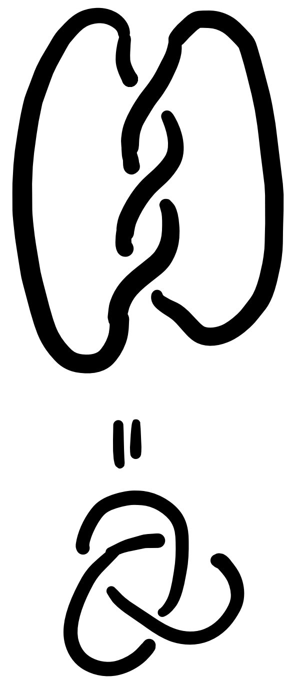
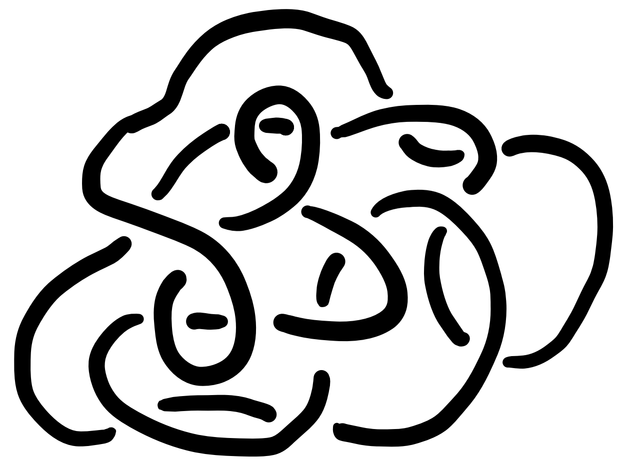


unknot



trefoil



KNOTS

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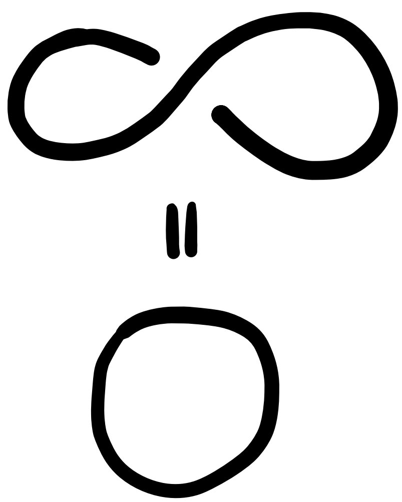
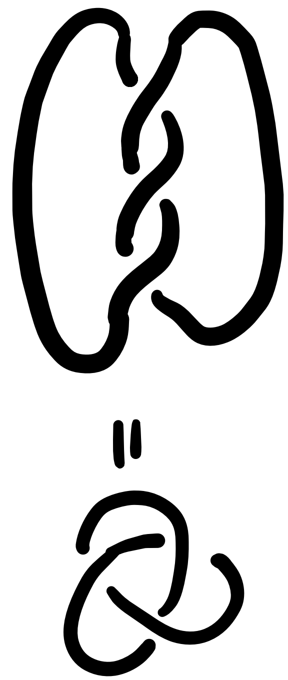
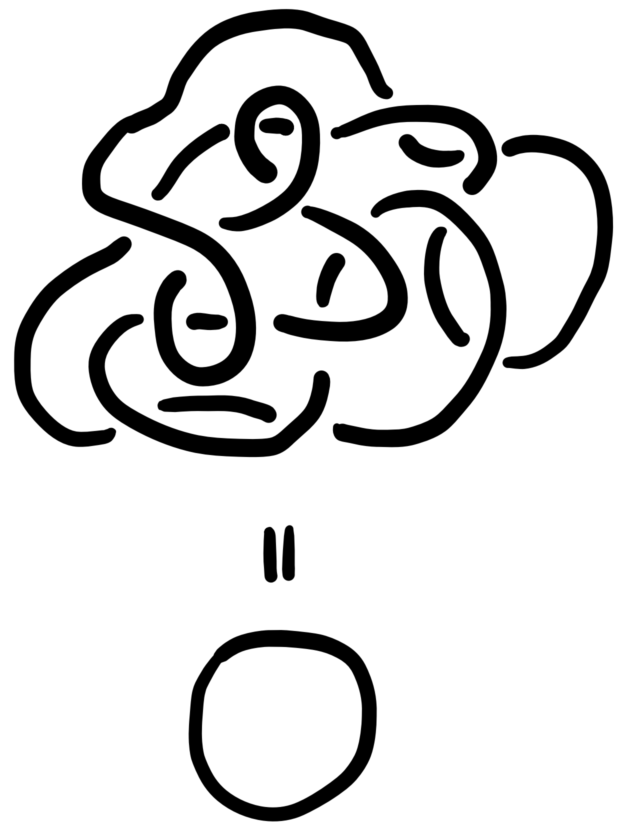


unknot



trefoil



unknot

KNOTS

(a) Motivating question: how can we tell when two knots are the same?

KNOTS

~ historical interlude ~

KNOTS

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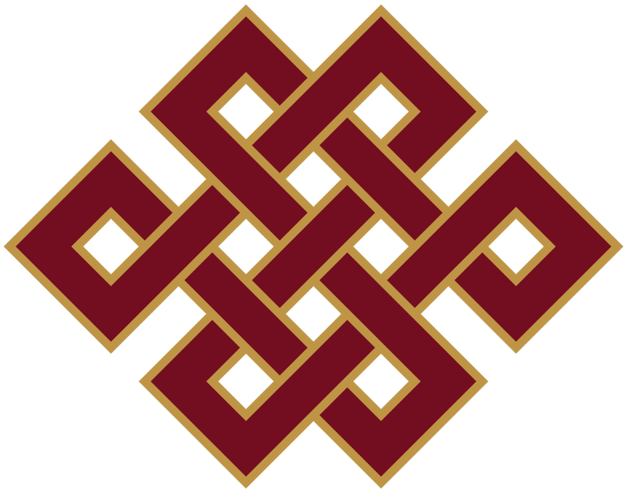
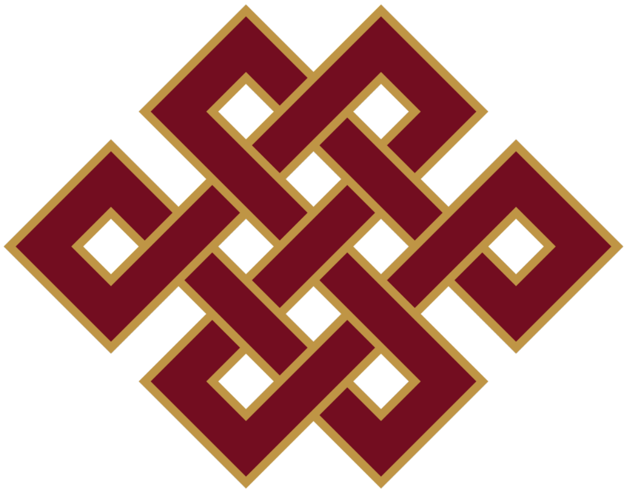
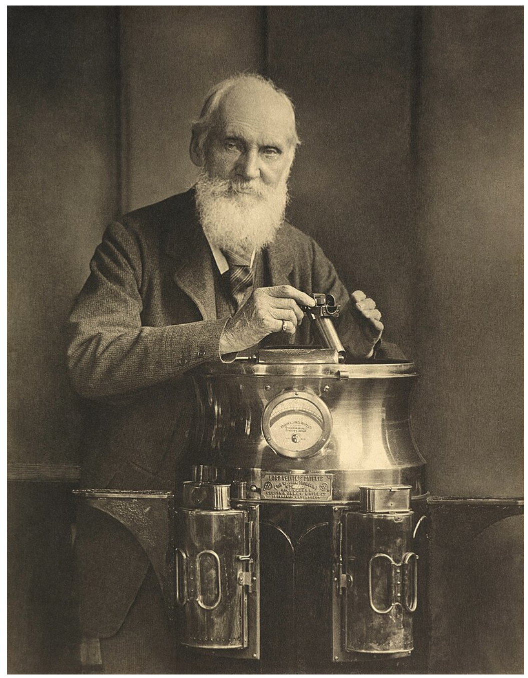


KNOTS

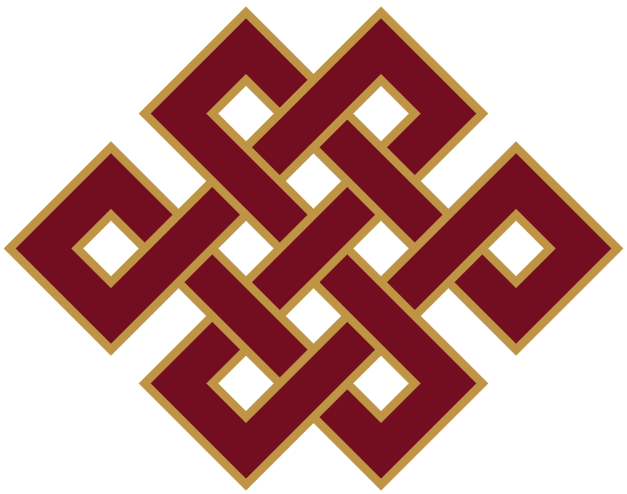
~ historical interlude ~

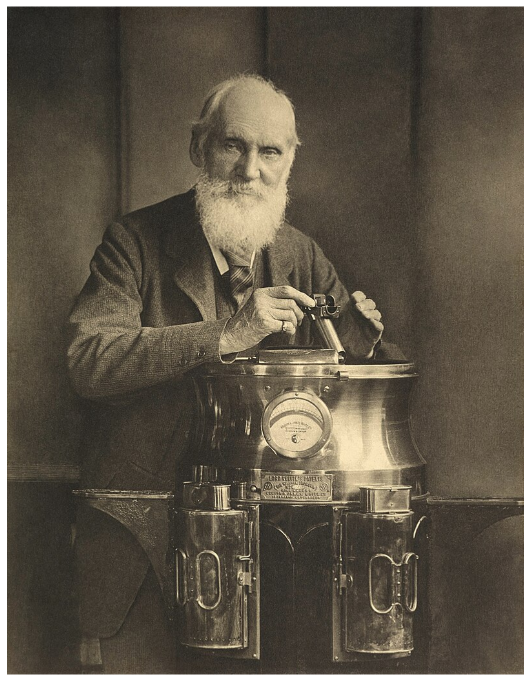


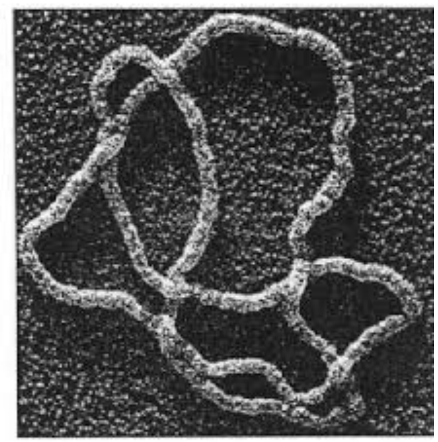


KNOTS

~ historical interlude ~







(a) Motivating question: how can we tell when two knots are the same?

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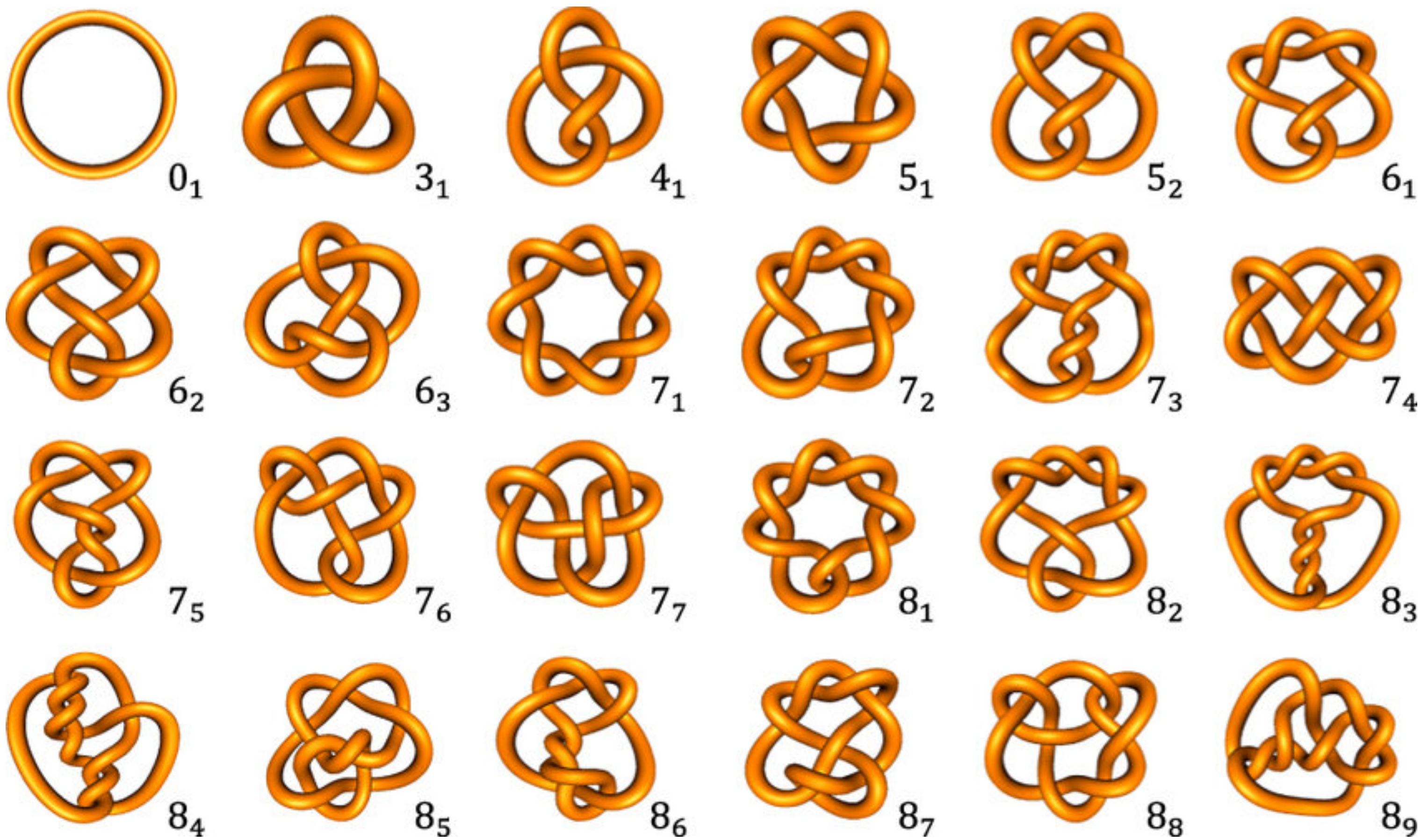
we can use knot invariants to help in this endeavor

INVARIANTS

e.g. Knots are classified via minimal crossing #, which is an invariant



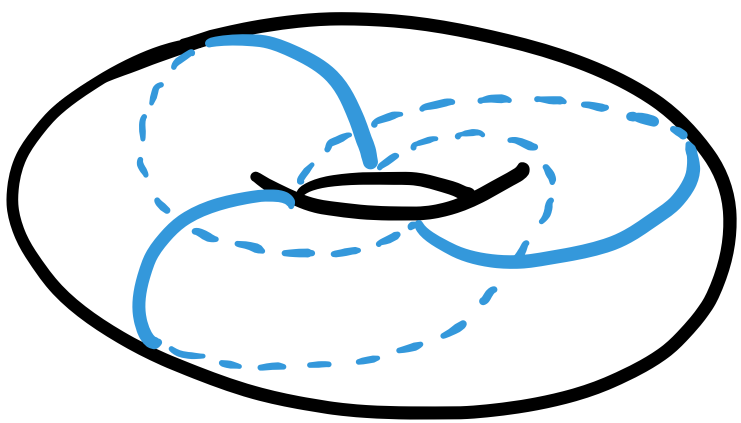
INVARIANTS

- crossing # $c(K)$
- unknotting # $u(K)$
- genus $g(K)$
- alexander polynomial $\Delta_t(K)$
- Jones polynomial $V_t(K)$
- fundamental group $\pi_1(S^3 \setminus K)$

TORUS KNOTS

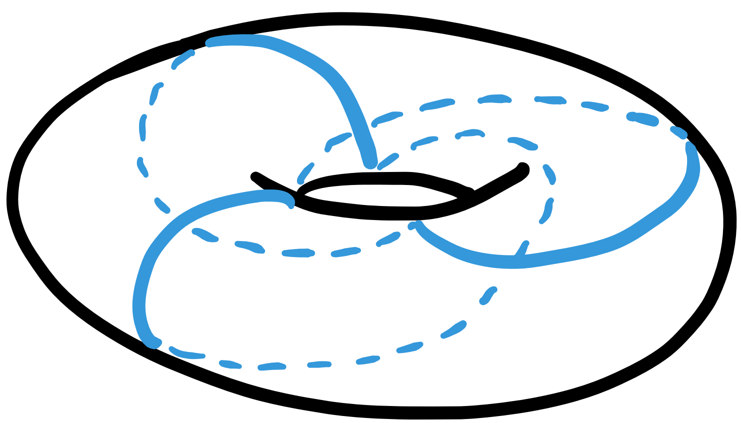
TORUS KNOTS

a **torus knot** is a knot which can sit on a torus
(w/o intersecting itself)



TORUS KNOTS

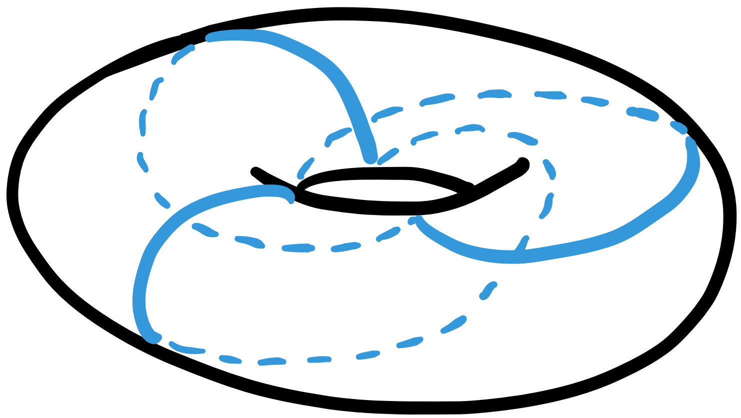
a torus knot is a knot which can sit on a torus
(w/o intersecting itself)



trefoil!

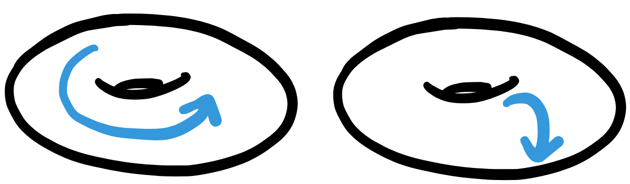
TORUS KNOTS

a **torus knot** is a knot which can sit on a torus (w/o intersecting itself)



trefoil! $= T(3,2)$

denote $T(p,q)$ for p,q rel. prime



TORUS KNOTS

some facts:

- $T(p,q) = T(q,p)$
- $c(T(p,q)) = \min((p-1)q, (q-1)p)$
- $u(T(p,q)) = \frac{1}{2}(p-1)(q-1)$
- $g(T(p,q)) = \frac{1}{2}(p-1)(q-1)$
- $\pi_1(S^3 \setminus T(p,q)) \cong \langle x,y \mid x^p = y^q \rangle$

BRAID FORM

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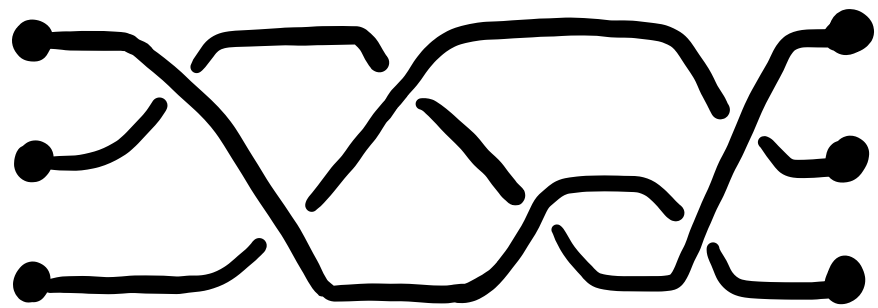
braid:



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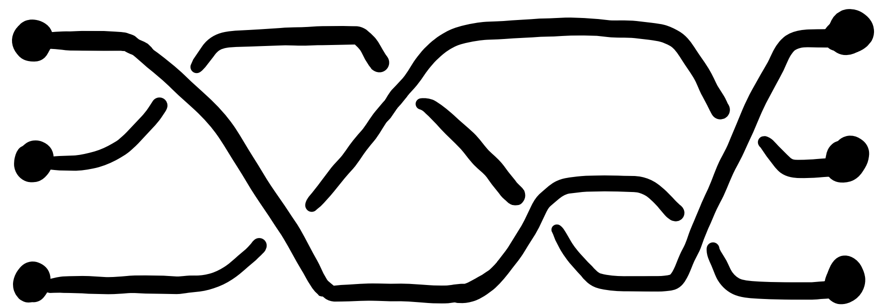
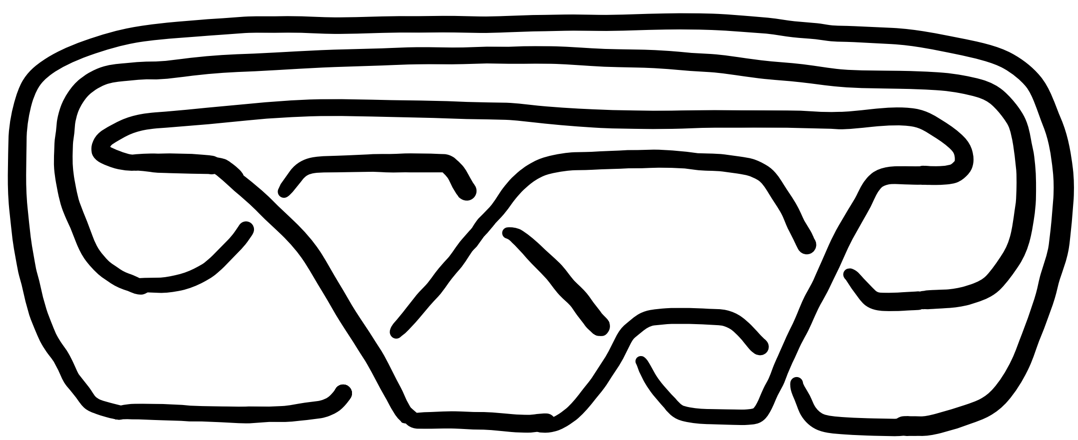
braid:

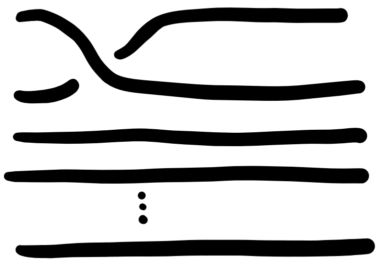


braid closure:
(= knot or "link")

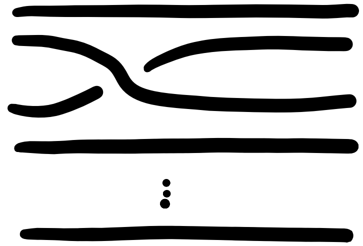


BRAID FORM

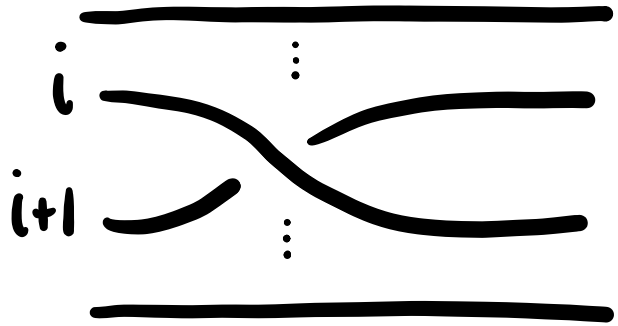
σ_1



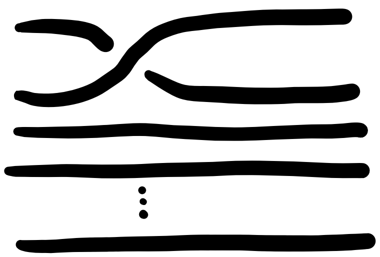
σ_2



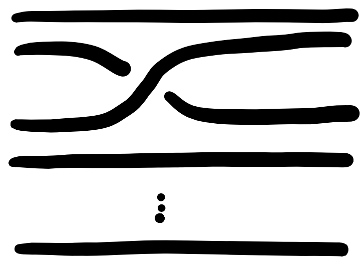
σ_i



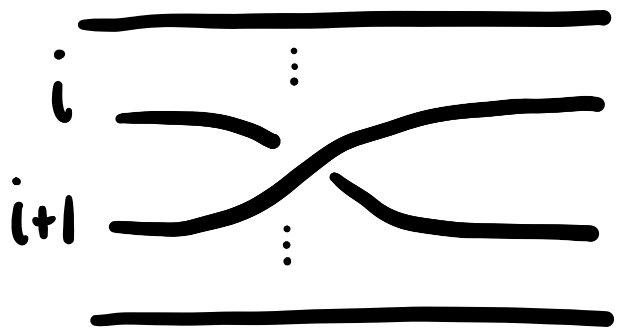
σ_1^{-1}



σ_2^{-1}



σ_i^{-1}



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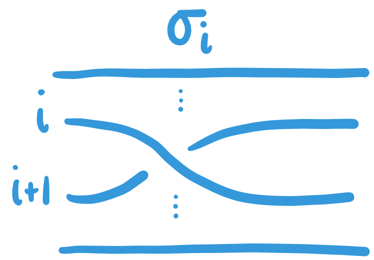

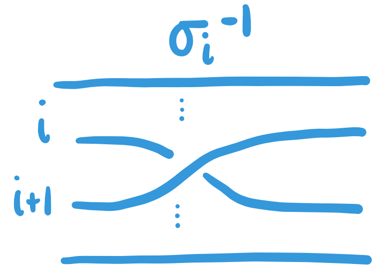

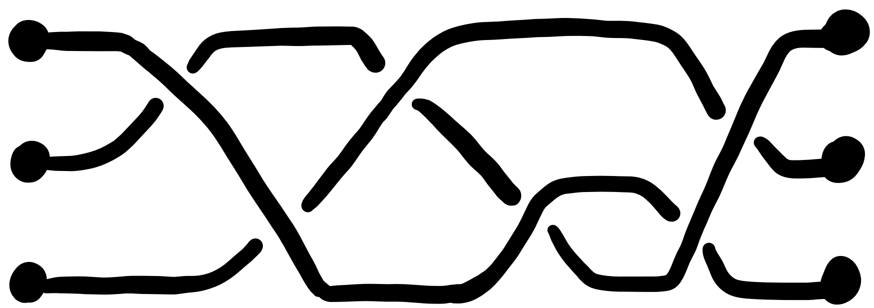

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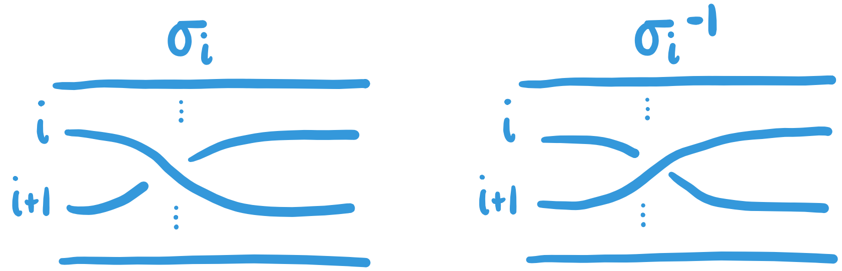
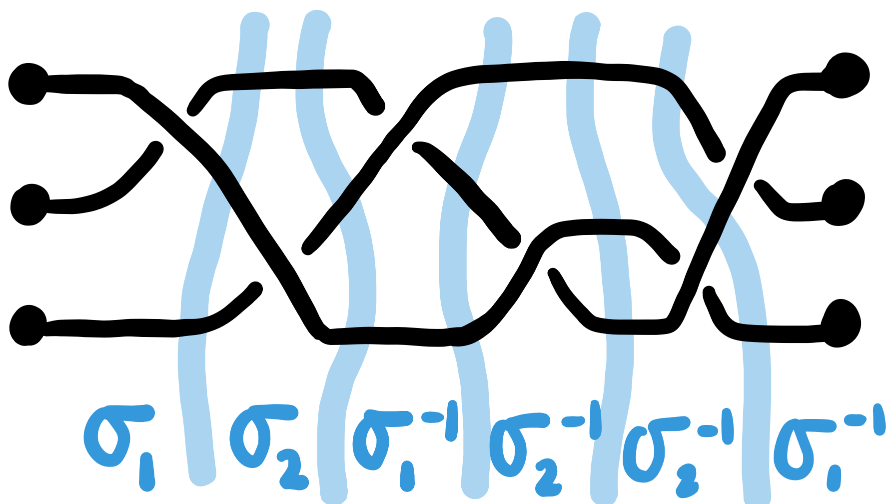


σ_i
i
$i+1$
σ_i^{-1}
i
$i+1$



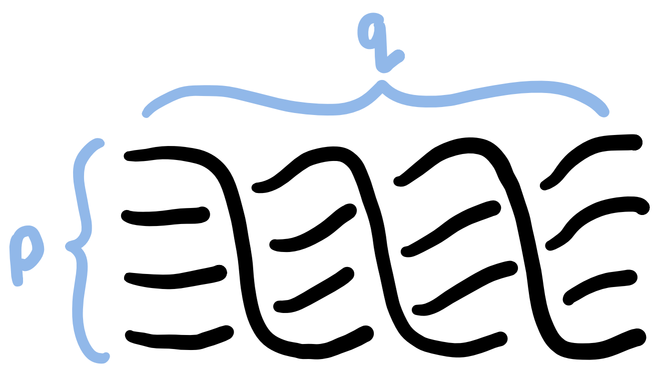
σ_1 σ_2 σ_1^{-1} σ_2^{-1} σ_2^{-1} σ_1^{-1}

TORUS KNOTS

$T(p,q)$

braid form:

$(\sigma_1 \sigma_2 \cdots \sigma_{p-1})^q$



$T(4,3)$

TORUS KNOTS

$T(p,q)$

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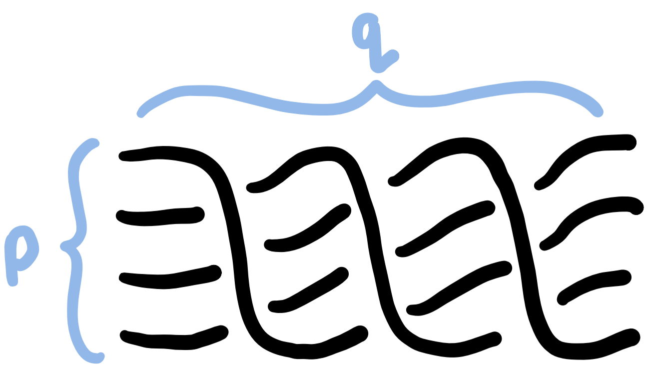

$T(4,3)$

SPIRAL KNOTS

$S(p,q,\varepsilon)$ $\quad \varepsilon = (\varepsilon_1, \dots, \varepsilon_{p-1})$

$\varepsilon_i \in \{\pm 1\}$

braid form:

$(\sigma_1^{\varepsilon_1} \sigma_2^{\varepsilon_2} \cdots \sigma_{p-1}^{\varepsilon_{p-1}})^q$

$S(4,3,(-1,1,-1))$

ε\q	2	3	4	5	6	7
(1,1)	3_1	–	8_{19}	10_{124}	–	X
(1,-1)	4_1	–	8_{18}	10_{123}	–	X
(1,1,1)	–	8_{19}	–	X	–	
(1,1,-1)	–	9_{47}	–	X	–	
(1,-1,1)	–	9_{40}	–	X	–	
(1,1,1,1)	5_1	10_{124}		–		
(1,1,1,-1)	6_2	$11n_{133}$		–		
(1,1,-1,1)	7_6	$12n_{839}$		–		
(1,1,-1,-1)	6_3	$12n_{706}$		–		
(1,-1,1,-1)	8_{12}	$12a_{1019}$		–		
(1,-1,-1,1)	7_7	$12n_{837}$		–		

Table 2: Spiral knots with low crossing number, tabulated by ε and q.

(1,1,1,1,1,1)	(1,1,1,1,1,-1)	(1,1,1,1,-1,1)	(1,1,1,1,-1,-1)	(1,1,1,-1,1,1)
7_1	8_2	9_{11}	8_7	9_{20}
(1,1,1,-1,1,-1)	(1,1,1,-1,-1,1)	(1,1,1,-1,-1,-1)	(1,1,-1,1,1,-1)	(1,1,-1,1,-1,1)
10_{29}	9_{26}	8_9	10_{44}	$11a_{159}$
(1,1,-1,1,-1,-1)	(1,1,-1,-1,1,1)	(1,1,-1,-1,1,-1)	(1,1,-1,-1,-1,1)	(1,-1,1,1,1,-1)
10_{43}	9_{31}	10_{42}	9_{27}	10_{41}
(1,-1,1,1,-1,1)	(1,-1,1,-1,1,-1)	(1,-1,1,-1,-1,1)	(1,-1,-1,1,1,-1)	(1,-1,-1,-1,-1,1)
$11a_{121}$	$12a_{477}$	$11a_{96}$	10_{45}	9_{17}

Table 3: Every spiral knot of the form $S(7, 2, \varepsilon)$, tabulated by ε.

	TORUS KNOTS	SPIRAL KNOTS	PERIODIC KNOTS

TORUS KNOTS $\subsetneq$ SPIRAL KNOTS $\subsetneq$ PERIODIC KNOTS

	TORUS KNOTS $\subsetneq$ SPIRAL KNOTS $\subsetneq$ PERIODIC KNOTS		
	TORUS KNOTS	SPIRAL KNOTS	PERIODIC KNOTS
Δ_K monic	YES	YES	NO

	TORUS KNOTS $\subsetneq$	SPIRAL KNOTS $\subsetneq$	PERIODIC KNOTS
Δ_K monic	YES	YES	NO
$g_3(K)$	$\frac{1}{2}(p-1)(q-1)$	$\frac{1}{2}(p-1)(q-1)$	Edmonds...

TORUS KNOTS $\subsetneq$ SPIRAL KNOTS $\subsetneq$ PERIODIC KNOTS

	TORUS KNOTS	SPIRAL KNOTS	PERIODIC KNOTS
Δ_K monic	YES	YES	NO
$g_3(K)$	$\frac{1}{2}(p-1)(q-1)$	$\frac{1}{2}(p-1)(q-1)$	Edmonds...
$g_4(K)$	$\neq 0$	could $=0$ (10_{123})	could $=0$

TORUS KNOTS $\subsetneq$ SPIRAL KNOTS $\subsetneq$ PERIODIC KNOTS

	TORUS KNOTS	SPIRAL KNOTS	PERIODIC KNOTS
Δ_K monic	YES	YES	NO
$g_3(K)$	$\frac{1}{2}(p-1)(q-1)$	$\frac{1}{2}(p-1)(q-1)$	Edmonds...
$g_4(K)$	$\neq 0$	could $=0$ (10_{123})	could $=0$
swap(p,q)	YES	ONLY for torus knots	—

	TORUS KNOTS $\subsetneq$	SPIRAL KNOTS $\subsetneq$	PERIODIC KNOTS
Δ_K monic	YES	YES	NO
$g_3(K)$	$\frac{1}{2}(p-1)(q-1)$	$\frac{1}{2}(p-1)(q-1)$	Edmonds...
$g_4(K)$	$\neq 0$	could $=0$ (10_{123})	could $=0$
swap(p,q)	YES	ONLY for torus knots	—
classified by Δ_K	YES	NO	NO

	TORUS KNOTS $\subsetneq$	SPIRAL KNOTS $\subsetneq$	PERIODIC KNOTS
Δ_K monic	YES	YES	NO
$g_3(K)$	$\frac{1}{2}(p-1)(q-1)$	$\frac{1}{2}(p-1)(q-1)$	Edmonds...
$g_4(K)$	$\neq 0$	could $=0$ (10_{123})	could $=0$
swap(p,q)	YES	ONLY for torus knots	—
classified by Δ_K	YES	NO	NO
classified by $\Delta_K + V_K$	YES	NO	NO

THANKS!