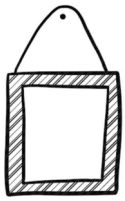
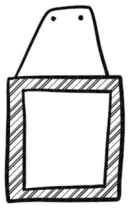
Warm-Up: Painting Hanging Problem

Can you hang a painting with two nails in such a way that if either nail breaks, the painting will fall?

Artie works at an art museum. His job is to hang up the paintings, which he has done by attaching a string to each, and hanging them with one nail, as shown below. (The string cannot be removed or changed once attached to the painting.)

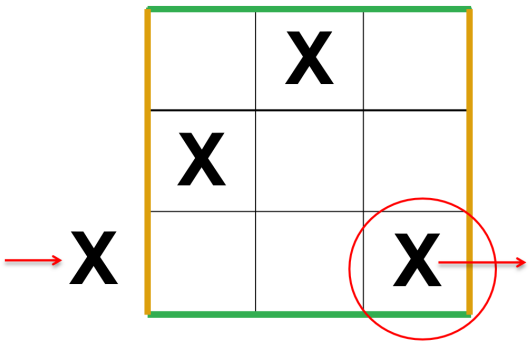


After he is finished, his boss exclaims that the paintings are too valuable to hang with only one nail, for if the nail breaks then the painting will fall and be damaged. His boss orders him to go back and hang each painting with two nails.



Artie is upset by this, because he will have to redo all of the work he has already done, which took him a really long time. So, out of spite, he finds a way to hang each painting with two nails, but such that if either nail breaks, the painting will fall. How did he do it?

Activity 1: Torus Tic-Tac-Toe



Question 0. Play a few games of Torus Tic-Tac-Toe with a neighbor.

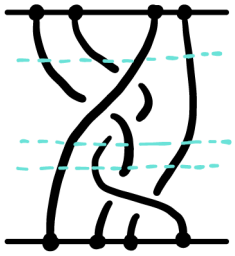
Question 1. What are all the new ways you can win in Torus Tic-Tac-Toe?

Question 2. Unlike traditional Tic-Tac-Toe, this game will never end in a draw. Why?

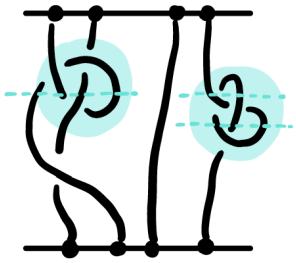
Question 3. What other games could you modify in a similar way (by gluing the sides of the board together)? How would this change these games?

Activity 2: Braids

A **n-strand braid** is a collection of n strings, attached to a horizontal bar at the top and bottom, with the rule that each string always heads downwards as we move along it from top to bottom.



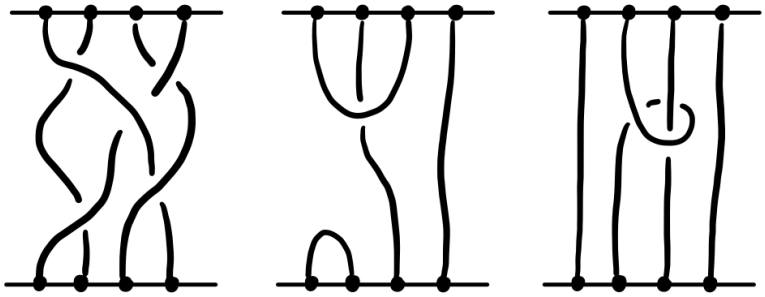
a braid!



not a braid :(

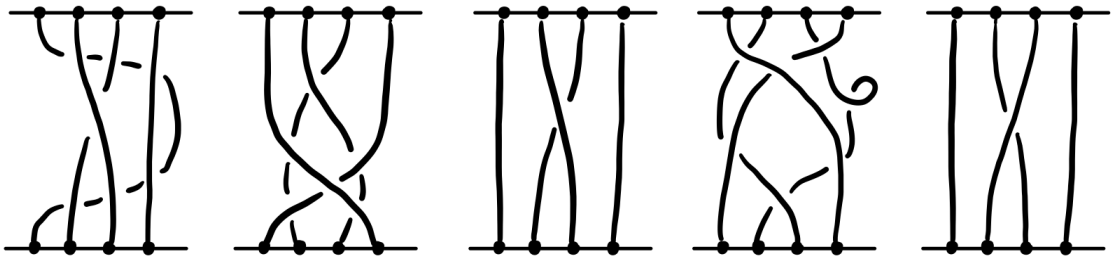
For a braid, it must be possible to wiggle the strings into a position so that any horizontal line intersects each string *exactly once*.

Question 1. Which of the following are braids?

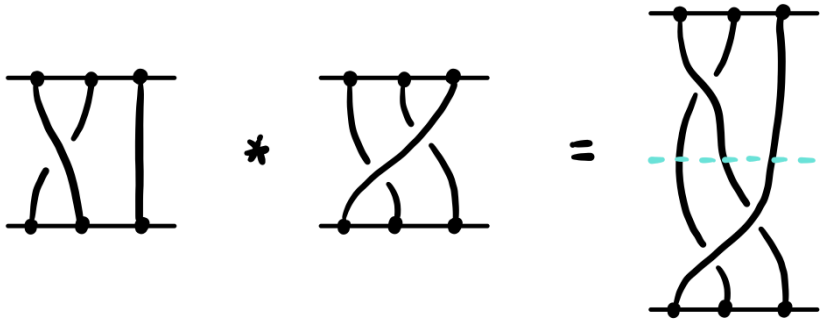


Question 2. Draw your own example of a braid.

Question 3. Two n-strand braids are the same if the strands of one can be wiggled to look like the other (without moving the endpoints). Which of the following braids are the same?

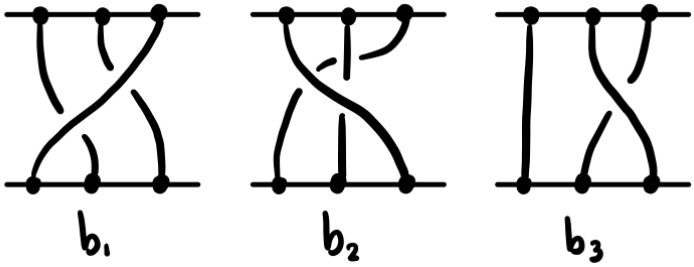


If we stack two n-strand braids on top of one another, we obtain a new braid! We can think of stacking as an operation that lets us "add" two braids together, and we will denote this operation with the symbol $*$. Note that we are only allowed to stack two braids if they have the same number of strands.

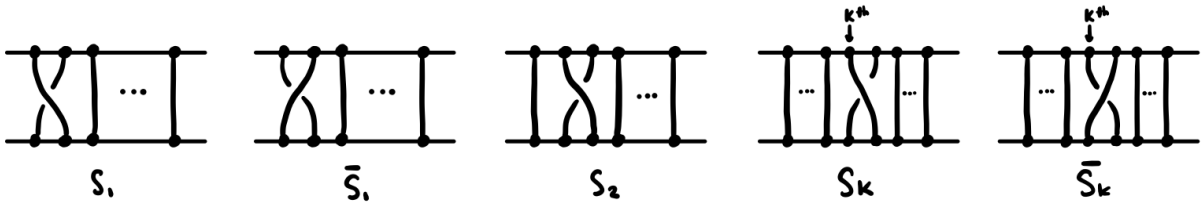


Question 4. Draw the results of stacking the following braids (shown below). Simplify your pictures as much as possible...

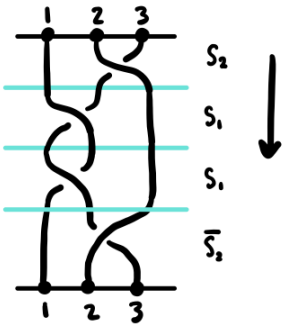
(a) $b_1 * b_2$, (b) $b_2 * b_3$, (c) $b_1 * b_3$



We can encode braids using **elementary braids**.



Every braid is made up of a collection of elementary braids stacked together vertically. So, we can represent a braid by a list of elementary braids separated by stacking symbols, which we call a **braid word**.



$s_2 * s_1 * s_1 * \overline{s_2}$

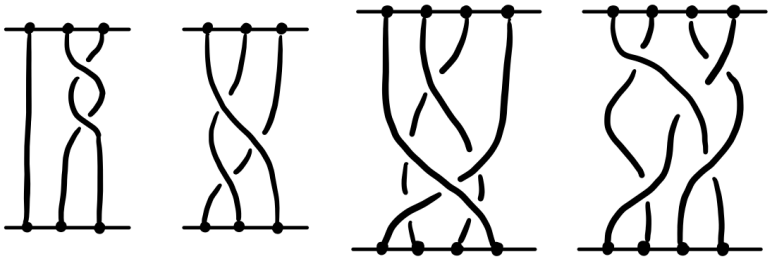
Question 5. Draw a picture of the braids represented by these braid words.

(a) The 3-stranded braid $s_1 * s_1 * s_2 * \overline{s_1} * \overline{s_2}$.

(b) The 4-stranded braid $s_1 * s_2 * s_3 * \overline{s_2}$.

(c) The 5-stranded braid $s_1 * s_2 * s_3 * \overline{s_2}$.

Question 6. Determine a braid word corresponding to each of these braids. Are braid words unique? That is, can two different braid words represent the same braid?



Challenge Questions.

(a) For a given n, is the set of all n-strand braids finite or infinite? Why?

(b) Does stacking order matter? That is, if b_1 and b_2 are two braids, is $b_1 * b_2$ always the same as $b_2 * b_1$?

(c) What happens if we stack s_k on top of $\overline{s_k}$?

(d) Can you find an n-strand braid e with the following property: for any other n-strand braid b, the braid $b * e$ is the same as b? (In other words, stacking by e doesn't change the braid.)

(e) Given a braid b, can you describe how to find a braid c such that $b * c = e$?