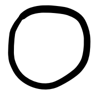
Sarah Blackwell

11/3/24

MATH CIRCLE: KNOT THEORY

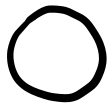
to create a mathematical knot: tie a knot in a piece of string and then glue the ends up
(so you have a knotted circle)

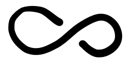
Q: are these the same?


unknot


trefoil

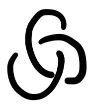
Q: are these the same?





both unknots!

if you can wiggle one knot to another without cutting, gluing, or passing the knot through itself, then they are the same. (Else, not.)

seems like  ≠  ...

how do we prove this??
(one of our goals today!)

(✱ a major goal/challenge in the study of knots is determining when two knots are the same)

short activity: (1) cut a piece of yarn and tie it into a figure-8 knot
(2) move the yarn around (so it looks different...) and lay on table
(3) draw (on paper) the result
(4) put result on the board

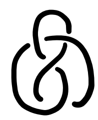
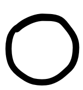
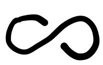


figure-8

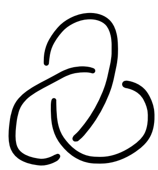
Q: observations about collection of diagrams?
(no wrong answers) (bring up crossing # - max? min?)

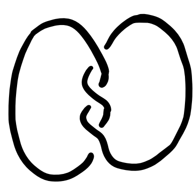
diagrams are not "unique":





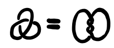
these are all the unknot



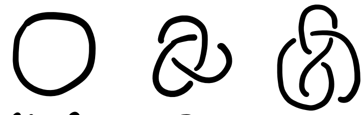


two diagrams for the trefoil

PROBLEMS

(1) how many knots are there with min. crossing # 1?
(2) convince yourself (with drawing or yarn) that 
(3) how many knots are there with min. crossing # 2? (harder)
(4) how many knots are there with min. crossing # 3? (hardest)

we classify knots by minimal crossing number :



U or 0_1 3_1 4_1

min. crossing # index

(show knot table)

crossing #	3	4	5	6	7	8	9	10	11	12	13	14	15	16
# (prime) knots	1	1	2	3	7	21	49	165	552	2176	9988	46,972	253,293	1,388,705

(POSSIBLE BREAK TIME)

TRICOLORABILITY

(one strategy for telling knots apart!)

a knot diagram is colorable if each arc in the diagram can be colored one of three colors, st:
1. at every crossing, either all strands are the same color or all are different
2. at least two colors are used

ex

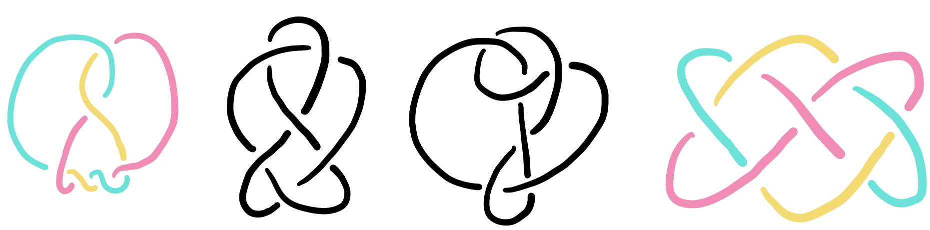
not colorable (violates rule 2)

not a valid coloring (violates rule 1)

a valid coloring ✓ so diagram is colorable

PROBLEMS

which diagrams are colorable?



6_1 ✓ 6_2 X 6_3 X 7_4 ✓

talk about how to prove these are not colorable!

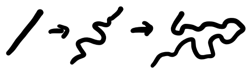
fact: for a given knot, either every diagram is colorable, or no diagram is colorable.
so, if you have two diagrams, st one is colorable and one is not... then the knots are not the same!

∴ the trefoil ≠ the unknot! one is colorable and one is not.

(POSSIBLE BREAK TIME) (roughly ~1hr)

MOVES ON DIAGRAMS

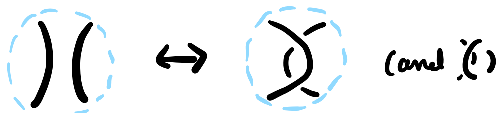
(0) wiggling within the plane
(not messing with crossings)

i.e. 

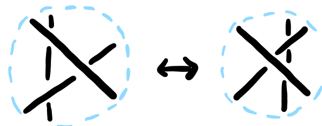
(1) Reidemeister I move.

local picture: everything outside this picture stays the same

(2) Reidemeister II move.



(3) Reidemeister III move.



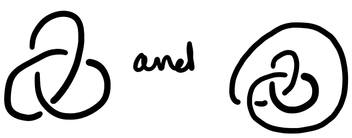
(or with strand over the crossing)

theorem: any two diagrams of the same knot can be related by a series of the above moves

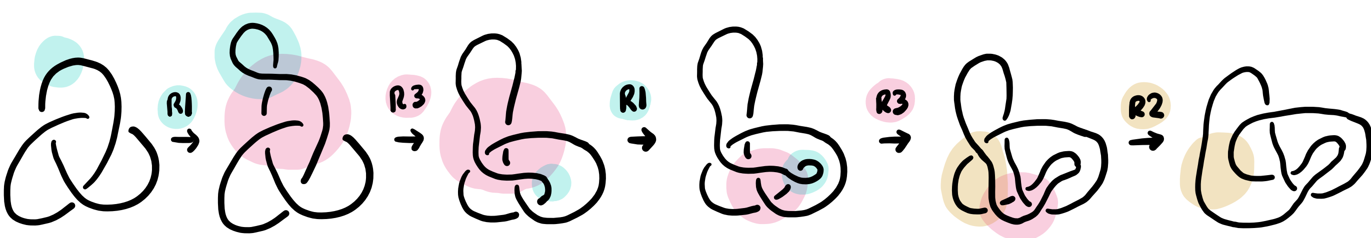
PROBLEMS

find a sequence of Reidemeister moves to go between...

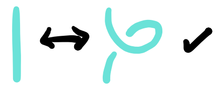
(1) 

(2) 

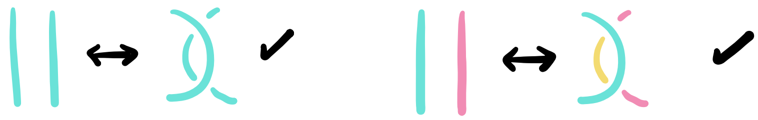
(3) 



(can do some planar isotopies to clean this up) we can prove that colorability can tell knots apart by showing that Reidemeister moves preserve colorability.

R1: only one case to check. 

R2: two cases to check. Either the two strands are the same color, or they aren't.



(given the left hand side, we can find a compatible coloring on the right hand side)

PROBLEM

do the R3 case!

POSSIBLE HW:

- problems above (not finished)
- more colorability
- more R-moves

if you draw a knot diagram st whenever you meet a part you drew previously, you always go under, you will always get an unknot. Why?